Moment bounds and central limit theorems for Gaussian subordinated arrays

Jean-Marc Bardet[1] and Donatas Surgailis[2,*]

bardet@univ-paris1.fr, donatas.surgailis@mii.vu.lt

[1] SAMM, Université Paris 1, 90 rue de Tolbiac, 75013 Paris, FRANCE

[2]Institute of Mathematics and Informatics, Vilnius University, Akademijos 4, 08663 Vilnius, LITHUANIA

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Abstract

A general moment bound for sums of products of Gaussian vector's functions extending the moment bound in Taqqu (1977, Lemma 4.5) is established. A general central limit theorem for triangular arrays of nonlinear functionals of multidimensional non-stationary Gaussian sequences is proved. This theorem extends the previous results of Breuer and Major (1981), Arcones (1994) and others. A Berry-Esseen-type bound in the above-mentioned central limit theorem is derived following Nourdin, Peccati and Podolskij (2011). Two applications of the above results are discussed. The first one refers to the asymptotic behavior of a roughness statistic for continuous-time Gaussian processes and the second one is a central limit theorem satisfied by long memory locally stationary process.

Keywords: Central limit theorem for triangular arrays; Moment bound for Gaussian vector's functions; Hermitian decomposition; Diagram formula; Berry-Esseen bounds; Long memory processes; Locally stationary process.

1 Introduction

This paper is devoted to the proof of two new results concerning functions of Gaussian vectors. The first one (Lemma 1 of Section 2) is a moment bound for "off-diagonal" sums of products of functions of Gaussian vectors in a general frame. It is an extension of an important lemma by Taqqu (1977, Lemma 4.5). This result is useful for obtaining almost sure convergence and tightness of Gaussian subordinated functionals and statistics, see Remark 1 below. The proof of Lemma 1 uses the Hermitian decomposition of $\mathbb{L}^2$ function and the diagram formula. A related but different moment bound is proved in Soulier (2001, Corollary 2.1).

The second result is a central limit theorem (CLT) for arrays of random variables that are functions of Gaussian vectors, see Theorem 1 for a precise statement. Theorem 1 generalizes and extends earlier results due to Breuer and Major (1983), Giraitis and Surgailis (1985) and Arcones (1994, Theorem 2) to the case of non-stationary triangular arrays of Gaussian vectors. Extensions of the Breuer-Major theorem were also obtained by Chambers and Slud (1989), Sanchez de Naranjo (1993) and Nourdin *et al.* (2011). Most of the above cited papers treat the case of a single stationary Gaussian sequence and a function independent

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of n. Generalization to stationary or non-stationary triangular arrays is motivated by numerous statistical applications. Some examples of these applications, with a particular emphasis on strongly dependent Gaussian processes, are: statistics of time series (see for instance Bardet *et al.*, 2008, Roueff and von Sachs, 2010), kernel-type estimation of regression function (Guo and Koul, 2008), nonparametric estimation of the local Hurst function of a continuous-time process from a discrete grid $i/n, 1 \leq i \leq n$ (Guyon and Leon, 1989, Bardet and Surgailis 2011, 2012). Two particular applications (limit theorems for the Increment Ratio statistic of a Gaussian process admitting a tangent process and a CLT for functions of locally stationary Gaussian process) are discussed in Section 5.

Starting with the famous Lindeberg Theorem for independent random variables, numerous studies devoted to CLT for triangular arrays under various dependence conditions had appeared. The case of martingale dependence was extensively studied in Jacod and Shiryaev (1987). Rio (1995) discussed the case of strongly mixing sequences. Some of more recent papers devoted to this question are Coulon-Prieur and Doukhan (2000) (with a new weak dependence condition) and Dedecker and Merlevède (2002) (with a necessary and sufficient condition for stable convergence of normalized partial sums). The CLT for linear triangular arrays was discussed in detail in Peligrad and Utev (1997) for several forms of dependence conditions.

The case of Gaussian subordinated variables (functions of Gaussian vectors) is rather exceptional among other dependence structures since it allows for very sharp conditions for CLT in terms of the decay rate of the covariance of Gaussian process and the Hermite rank of non-linear function. These conditions are close to being necessary and result in CLTs "in the vicinity" of non-central limit theorems, see Breuer and Major (1981), Arcones (1994), Dobrushin and Major (1979), Taqqu (1979). The proofs of the above-mentioned results rely on specific Gaussian techniques such as the Hermite expansion and the diagram formula; however, the recent paper Nourdin *et al.* (2011) uses a different approach based on Malliavin's calculus and Stein's method, yielding also convergence rates in the CLT. The main difference between our Theorem 1 and the corresponding results in Arcones (1994) and Nourdin *et al.* (2011) is that, contrary to these papers, we do not assume stationarity of the underlying Gaussian sequence $(\boldsymbol{Y}_n(k))$ and discuss the case of subordinated sums $\sum_{k=1}^n f_{k,n}(\boldsymbol{Y}_n(k))$ where $f_{k,n}$ may depend on k and n. The last fact is important for statistical applications (see above). In the particular case when $f_{k,n} = f$ do not depend on k, n and $(\boldsymbol{Y}_n(k))$ is a stationary process independent of n, Theorem 1 (iii) agrees with Arcones (1994) and Nourdin *et al.* (2011, Theorem 1.1). The proof of Theorem 1 uses the diagram method and cumulants as in Giraitis and Surgailis (1985). Section 4 obtains a Berry-Esseen bound in this CLT using the approach and results in Nourdin *et al.* (2011). Let us note that a CLT for Gaussian subordinated arrays is also proved in Soulier (2001, Theorem 3.1); however, it requires that Gaussian vectors are asymptotically independent and therefore his result is different from Theorem 1.

Notation. Everywhere below, $\boldsymbol{X} = (X^{(1)}, \ldots, X^{(\nu)})$ designates a standardized Gaussian vector in $\mathbb{R}^\nu$, $\nu \geq 1$, with zero mean $\mathrm{E}X^{(u)} = 0$ and covariances $\mathrm{E}X^{(u)}X^{(v)} = \delta_{uv}$, $u, v = 1, \ldots, \nu$. Letter C stands for a constant whose precise value is unimportant and which may change from line to line. The weak convergence of distributions is denoted by $\underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}$.

2 A moment bound

Let $\mathbb{L}^2(\boldsymbol{X})$ denote the class of all measurable functions $f = f(\boldsymbol{x}), \boldsymbol{x} = (x^{(1)}, \ldots, x^{(\nu)}) \in \mathbb{R}^\nu$ such that $\|f\|^2 := \mathrm{E}f^2(\boldsymbol{X}) < \infty$. For any multiindex $\boldsymbol{k} = (k^{(1)}, \ldots, k^{(\nu)}) \in \mathbb{Z}_+^\nu := \{(j^{(1)}, \ldots, j^{(\nu)}) \in \mathbb{Z}^\nu, j^{(u)} \geq 0 \ (1 \leq u \leq \nu)\}$, let $H_{\boldsymbol{k}}(\boldsymbol{x}) = H_{k^{(1)}}(x^{(1)}) \cdots H_{k^{(\nu)}}(x^{(\nu)})$ be the (product) Hermite polynomial; $H_k(x) := (-1)^k \mathrm{e}^{x^2/2}(\mathrm{e}^{-x^2/2})^{(k)}$, $k = 0, 1, \ldots$ are standard Hermite polynomials (with $(\mathrm{e}^{-x^2/2})^{(k)}$ the kth derivative of the function $x \mapsto \mathrm{e}^{-x^2/2}$). Write $|\boldsymbol{k}| := k^{(1)} + \ldots + k^{(\nu)}$, $\boldsymbol{k}! := k^{(1)}! \cdots k^{(\nu)}!$, $\boldsymbol{k} = (k^{(1)}, \ldots, k^{(\nu)}) \in \mathbb{Z}_+^\nu$. A function $f \in \mathbb{L}^2(\boldsymbol{X})$ is said to have a Hermite rank $m \geq 0$ if $J_f(\boldsymbol{k}) := \mathrm{E} f(\boldsymbol{X}) H_{\boldsymbol{k}}(\boldsymbol{X}) = 0$ for any $\boldsymbol{k} \in \mathbb{Z}_+^\nu, |\boldsymbol{k}| < m$, and $J_f(\boldsymbol{k}) \neq 0$ for some $\boldsymbol{k}, |\boldsymbol{k}| = m$. It is well-known that any $f \in \mathbb{L}^2(\boldsymbol{X})$ having a Hermite rank $m \geq 0$ admits the Hermite expansion

$$f(\boldsymbol{x}) = \sum_{|\boldsymbol{k}| \geq m} \frac{J_f(\boldsymbol{k})}{\boldsymbol{k}!} H_{\boldsymbol{k}}(\boldsymbol{x}), \tag{2.1}$$

which converges in $\mathbb{L}^2(\boldsymbol{X})$.

Let $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ be a collection of standardized Gaussian vectors $\boldsymbol{X}_t = (X_t^{(1)}, \ldots, X_t^{(\nu)}) \in \mathbb{R}^\nu$ having a joint Gaussian distribution in $\mathbb{R}^{\nu n}$. Let $\varepsilon \in [0,1]$ be a fixed number. Following Taqqu (1977), we call $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ $\varepsilon-$*standard* if $|\mathrm{E} X_t^{(u)} X_s^{(v)}| \leq \varepsilon$ for any $t \neq s, 1 \leq t, s \leq n$ and any $1 \leq u, v \leq \nu$.

As mentioned in the Introduction, Lemma 1 generalizes Taqqu (1977, Lemma 4.5) to the case of a vector-valued Gaussian family $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$, taking values in $\mathbb{R}^\nu (\nu \geq 1)$. The lemma concerns the bound (2.4), below, where $f_{1,t,n}, \ldots, f_{p,t,n}$ are square integrable functions among which the first $0 \leq \alpha \leq p$ functions $f_{1,t,n}, \ldots, f_{\alpha,t,n}$ for any $1 \leq t \leq n$ have a Hermite rank at least equal to $m \geq 1$ and where $\sum'$ is the sum over all different indices $1 \leq t_i \leq n$ $(1 \leq i \leq p), t_i \neq t_j (i \neq j)$. In the case when $f_{j,t,n} = f_j$ does not depend on t, n, the bound (2.4) coincides with that of Taqqu (1977, Lemma 4.5) provided $m\alpha$ is even, but is worse than Taqqu's bound in the more delicate case when $m\alpha$ is odd. An advantage of our proof is its relative simplicity (we do not use the graph-theoretical argument as in Taqqu, 1977, but rather a simple Hölder inequality). A different approach towards moment inequalities for functions in vector-valued Gaussian variables is discussed in Soulier (2001), leading to a different type of moment inequalities.

Lemma 1 *Let $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ be a $\varepsilon-$standard Gaussian vector, $\boldsymbol{X}_t = (X_t^{(1)}, \ldots, X_t^{(\nu)}) \in \mathbb{R}^\nu$, $\nu \geq 1$, and let $f_{j,t,n} \in \mathbb{L}^2(\boldsymbol{X}), 1 \leq j \leq p$, $p \geq 2$, $1 \leq t \leq n$ be some functions. For given integers $m \geq 1, 0 \leq \alpha \leq p, n \geq 1$, define*

$$Q_n \quad := \quad \max_{1 \leq t \leq n} \sum_{1 \leq s \leq n, s \neq t} \max_{1 \leq u, v \leq \nu} |\mathrm{E} X_t^{(u)} X_s^{(v)}|^m. \tag{2.2}$$

Assume that the functions $f_{1,t,n}, \ldots, f_{\alpha,t,n}$ have a Hermite rank at least equal to m for any $n \geq 1, 1 \leq t \leq n$, and that

$$\varepsilon < \frac{1}{\nu p - 1}. \tag{2.3}$$

Then

$$\sum{}' \left|\mathrm{E}\big[f_{1,t_1,n}(\boldsymbol{X}_{t_1}) \cdots f_{p,t_p,n}(\boldsymbol{X}_{t_p})\big]\right| \quad \leq \quad C(\varepsilon, p, m, \alpha, \nu) K n^{p - \frac{\alpha}{2}} Q_n^{\frac{\alpha}{2}}, \tag{2.4}$$

where the constant $C(\varepsilon, p, m, \alpha, \nu)$ depends on $\varepsilon, p, m, \alpha, \nu$ only, and

$$K = \prod_{j=1}^p \max_{1 \leq t \leq n} \|f_{j,t,n}\| \quad \textit{with} \quad \|f_{j,t,n}\|^2 = \mathrm{E}\big[f_{j,t,n}^2(\boldsymbol{X})\big]. \tag{2.5}$$

Proof. Fix a collection $(t_1, \ldots, t_p)$ of disjoint indices $t_i \neq t_j (i \neq j)$, and write $f_j = f_{j,t_j,n}$, $1 \leq j \leq p$ for brevity. Let $J_j(\boldsymbol{k}) := J_{f_j}(\boldsymbol{k}) = \mathrm{E}\big[f_j(\boldsymbol{X}) H_{\boldsymbol{k}}(\boldsymbol{X})\big]$ be the coefficients of the Hermite expansion of f_j. Then,

$$\begin{aligned} |J_j(\boldsymbol{k})| \quad &\leq \quad \|f_j\| \prod_{i=1}^\nu \mathrm{E}^{1/2} H_{k^{(i)}}^2(X) \\ &\leq \quad \|f_j\| \prod_{i=1}^\nu (k^{(i)}!)^{1/2} = \|f_j\| (\boldsymbol{k}!)^{1/2}. \end{aligned}$$

Following Taqqu (1977, p. 213, bottom, p. 214, top), we obtain

$$\begin{aligned}
|\mathrm{E} f_1(\boldsymbol{X}_{t_1})\cdots f_p(\boldsymbol{X}_{t_p})| &= \left|\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}\left\{\prod_{j=1}^{p}\frac{J_j(\boldsymbol{k}_j)}{\boldsymbol{k}_j!}\right\}\mathrm{E}\big[H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})\big]\right| \\
&\le K_1\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}\frac{|\mathrm{E} H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})|}{(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}} \\
&\le K_1\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}\frac{\varepsilon^{(|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|)/2}\mathrm{E}\prod_{1\le u\le\nu}\prod_{1\le j\le p}H_{k_j^{(u)}}(X)}{(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}} \\
&\le K_1\sum_{q=0}^{\infty}\sum_{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q}(\varepsilon(\nu p-1))^{(|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|)/2}<\infty,
\end{aligned}$$

where $X\sim\mathcal{N}(0,1)$ and

$$K_1 := \|f_{1,t_1,n}\|\cdots\|f_{p,t_p,n}\| \;\le\; K,$$

where K is defined in (2.5) and K is independent of $t_1,\ldots,t_p$, and where we used the assumption (2.3) to get the convergence of the last series. Therefore,

$$\sum\nolimits' \big|\mathrm{E}\big[f_{1,t_1,n}(\boldsymbol{X}_{t_1})\cdots f_{p,t_p,n}(\boldsymbol{X}_{t_p})\big]\big| \;\le\; K\sum_{q=0}^{\infty}\sum_{\substack{|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q\\ |\boldsymbol{k}_1|\ge m,\ldots,|\boldsymbol{k}_\alpha|\ge m}}\sum\nolimits'\frac{|\mathrm{E} H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})|}{(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}}.$$

Now, the following bound remains to be proved: for any integers $m\ge1, 0\le\alpha\le p, n\ge1$ and any multiindices $\boldsymbol{k}_1,\ldots\boldsymbol{k}_p\in\mathbb{Z}_+^\nu$ satisfying $|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|=2q,\ |\boldsymbol{k}_1|\ge m,\ldots,|\boldsymbol{k}_\alpha|\ge m$,

$$\sum\nolimits' |\mathrm{E} H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})| \;\le\; C_1(\varepsilon(\nu p-1))^{(|\boldsymbol{k}_1|+\ldots+|\boldsymbol{k}_p|)/2}(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2}n^{p-\frac{\alpha}{2}}Q_n^{\frac{\alpha}{2}}, \tag{2.6}$$

where C_1 is some constant depending only on p,ν,α,ε, and independent of $\boldsymbol{k}_1,\ldots,\boldsymbol{k}_p,n$.

First, we write the expectation on the left hand side of (2.6) as a sum of contributions of diagrams. Let

$$T := \begin{pmatrix} (1,1) & (1,2) & \ldots & (1,k_1)\\ (2,1) & (2,2) & \ldots & (1,k_2)\\ \ldots & & & \\ (p,1) & (p,2) & \ldots & (p,k_p)\end{pmatrix} \tag{2.7}$$

be a table having p rows $\tau_1,\ldots,\tau_p$ of respective lengths $|\tau_u|=k_u=|\boldsymbol{k}_u|=k_u^{(1)}+\ldots+k_u^{(\nu)}$ (we write $T=\bigcup_{u=1}^p\tau_u$). A *sub-table* of T is a table $T'=\bigcup_{u\in U}\tau_u,\ U\subset\{1,\ldots,p\}$ consisting of some rows of T written from top to bottom in the same order as rows in T; clearly any sub-table T' of T can be identified with a (nonempty) subset $U\subset\{1,\ldots,p\}$. A *diagram* is a partition γ of the table T by pairs (called *edges* of the diagram) such that no pair belongs to the same row. A diagram γ is called *connected* if the table T cannot be written as a union $T=T'\cup T''$ of two disjoint sub-tables T',T'' so that T' and T'' are partitioned by γ separately. Write $\Gamma(T),\Gamma_c(T)$ for the class of all diagrams and the class of all connected diagrams over the table T, respectively. Let

$$\rho(t,s) := \max_{1\le u,v\le\nu}|\mathrm{E}X_t^{(u)}X_s^{(v)}| \qquad (t\ne s).$$

Note $0\le\rho(t,s)\le\varepsilon$ and $Q_n=\max_{1\le t\le n}\sum_{1\le s\le n,s\ne t}\rho^m(t,s)$. By the diagram formula for moments of Hermite (Wick) polynomials (see e.g. Surgailis, 2000),

$$\begin{aligned}
|\mathrm{E} H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})| &\le \sum_{\gamma\in\Gamma(T)}\prod_{1\le u<v\le p}(\rho(t_u,t_v))^{\ell_{uv}} &(2.8)\\
&= \sum_{(U_1,\ldots,U_h)}\prod_{r=1}^{h}\sum_{\gamma\in\Gamma_c(U_r)}\prod_{u,v\in U_r,u<v}(\rho(t_u,t_v))^{\ell_{uv}}, &(2.9)
\end{aligned}$$

where ℓ_{uv} is the number of edges between rows τ_u and τ_v in the diagram γ over table T, and the sum $\sum_{(U_1,\dots,U_h)}$ is taken over all partitions $(U_1,\dots,U_h), h=1,2,\dots,[p/2]$ of $\{1,\dots,p\}$ by nonempty subsets U_r of cardinality $|U_r|\ge 2$. (Thus, (2.9) follows from (2.8) by decomposing $\gamma\in\Gamma(T)$ into connected components $\gamma_r\in\Gamma_c(U_r), r=1,\dots,h; h=1,\dots,[p/2]$; the restriction $|U_r|\ge 2$ stems from the fact that any edge must necessarily connect different rows.) From (2.9) we obtain

$$\sum{}' \left|\mathrm{E}\big[f_1(\boldsymbol{X}_{t_1})\cdots f_p(\boldsymbol{X}_{t_p})\big]\right| \;\le\; \sum_{(U_1,\dots,U_h)}\prod_{r=1}^{h}\sum_{\gamma\in\Gamma_c(U_r)} I_{n,U_r}(\gamma), \tag{2.10}$$

where, for any sub-table $U\subset T$ having at least two rows and for any connected diagram $\gamma\in\Gamma_c(U)$, the quantity $I_{n,U}(\gamma)$ is defined by

$$I_{n,U}(\gamma) := \sum{}' \prod_{u,v\in U, u<v} (\rho(t_u,t_v))^{\ell_{uv}}$$

where (recall) the product is taken over all ordered pairs of rows $(\tau_u,\tau_v), u<v$ of the table U, and ℓ_{uv} is the number of edges in γ between the uth and the vth rows. Below we prove the bound

$$I_{n,U}(\gamma)\le K_3\epsilon^{|\boldsymbol{k}_U|/2} n^{|U|-\frac{\alpha(U)}{2}}(nQ_n)^{\frac{\alpha(U)}{2}}, \tag{2.11}$$

where $|\boldsymbol{k}_U| := \sum_{u\in U}k_u$ is the number of points of table U and $\alpha(U) := |\{1,\dots,\alpha\}\bigcap U| = \#\{u\in U: |\boldsymbol{k}_u|\ge m\}$ is the number of rows in U having at least m points. Clearly, it suffices to show (2.11) for $U=T$.

Next, let for $1\le u,v\le p, u\ne v$, denote

$$R_{uv} \;:=\; \left(\sum\nolimits_{1\le t\le n}\left(\sum\nolimits_{1\le s\le n, s\ne t}\rho^{k_u}(s,t)\right)^{k_v/k_u}\right)^{\ell_{uv}/k_v}. \tag{2.12}$$

Let $A:=\{1,\dots,\alpha\}, A' := \{1,\dots,p\}\backslash A = \{\alpha+1,\dots,p\}$. It follows immediately from the definition of R_{uv} and $\rho(s,t)$ that

$$R_{uv} \;\le\; \begin{cases} n^{\frac{\ell_{uv}}{k_v}} Q_n^{\frac{\ell_{uv}}{k_u}}\varepsilon^{(1-\frac{m}{k_u})\ell_{uv}}, & \text{if } u\in A,\\ n^{\frac{\ell_{uv}}{k_u}+\frac{\ell_{uv}}{k_v}}\varepsilon^{\ell_{uv}}, & \text{if } u\in A^c.\end{cases} \tag{2.13}$$

By the Hölder inequality (see Giraitis and Surgailis, 1985, p.202, for details),

$$I_{n,T}(\gamma) \;\le\; \min\left(\prod_{1\le u<v\le p}R_{uv}, \prod_{1\le u<v\le p}R_{vu}\right). \tag{2.14}$$

For any subset $U\subset\{1,\dots,p\}$, let

$$L(U) := \sum_{u\in U}\sum_{u<v\le p}\frac{\ell_{uv}}{k_u}, \qquad L^*(U) := \sum_{u\in U}\sum_{1\le v<u}\frac{\ell_{uv}}{k_u}, \tag{2.15}$$

$L:=L(T), L^* := L^*(T)$. Clearly,

$$L(U)+L^*(U) \;=\; \sum_{u\in U}\frac{1}{k_u}\sum_{v=1,\dots,p, v\ne u}\ell_{uv} \;=\; |U| \tag{2.16}$$

is the number of points in U. From (2.13) - (2.14),

$$I_{n,T}(\gamma) \;\le\; \min\left(n^{L^*+L(A^c)}Q_n^{L(A)}\varepsilon^{|T|/2-mL(A)}, n^{L+L^*(A^c)}Q_n^{L^*(A)}\varepsilon^{|T|/2-mL^*(A)}\right),$$

where $|T| = \sum_{u=1}^p k_i$. As $0\le L(A), L^*(A)\le p$, see (2.16), we obtain

$$\begin{aligned} I_{n,T}(\gamma) \;&\le\; \varepsilon^{|T|/2-mp}\min\left(n^{L^*(A)+L^*(A^c)+L(A^c)}Q_n^{L(A)}, n^{L(A)+L(A^c)+L^*(A^c)}Q_n^{L^*(A)}\right)\\ &=\; \varepsilon^{|T|/2-mp}n^{p-\alpha}\min\left(n^{L^*(A)}Q_n^{L(A)}, n^{L(A)}Q_n^{L^*(A)}\right)\\ &=\; \varepsilon^{|T|/2-mp}n^{p-\alpha}(nQ_n)^{\frac{\alpha}{2}}\min\left((n/Q_n)^{\frac{\alpha}{2}-L(A)}, (n/Q_n)^{L(A)-\frac{\alpha}{2}}\right)\\ &\le\; \varepsilon^{|T|/2-mp}n^{p-\alpha}(nQ_n)^{\frac{\alpha}{2}}, \end{aligned}$$

proving (2.11).

With (2.11)-(2.10) in mind,

$$\begin{aligned}\sum{}' |\mathrm{E}H_{\boldsymbol{k}_1}(\boldsymbol{X}_{t_1})\cdots H_{\boldsymbol{k}_p}(\boldsymbol{X}_{t_p})| &\le C_3\varepsilon^{|T|/2}\sum_{(U_1,\dots,U_h)}\prod_{r=1}^{h}\sum_{\gamma\in\Gamma_c(U_r)} n^{|U_r|-\frac{\alpha(U_r)}{2}}Q_n^{\frac{\alpha(U_r)}{2}}\\ &= C_3\varepsilon^{|T|/2}n^{p-\frac{\alpha}{2}}Q_n^{\frac{\alpha}{2}}\sum_{(U_1,\dots,U_h)}\prod_{r=1}^{h}\sum_{\gamma\in\Gamma_c(U_r)} 1\\ &= C_3\varepsilon^{|T|/2}n^{p-\frac{\alpha}{2}}Q_n^{\frac{\alpha}{2}}\sum_{\gamma\in\Gamma(T)}1,\end{aligned}$$

where the last sum (= the number of all diagrams over the table T) does not exceed

$$|\mathrm{E}H_{k_1^{(1)}}(X)\cdots H_{k_1^{(\nu)}}(X)\cdots H_{k_p^{(1)}}(X)\cdots H_{k_p^{(\nu)}}(X)| \le (p\nu-1)^{(|\boldsymbol{k}_1|+\dots+|\boldsymbol{k}_p|)/2}(\boldsymbol{k}_1!\cdots\boldsymbol{k}_p!)^{1/2},$$

see Taqqu (1977, Lemma 3.1). This proves the bound (2.6) and the lemma, too. □

Lemma 1 can be extended to non-standardized Gaussians as follows. To this end, we introduce some definitions. Let $\boldsymbol{Y} = (Y^{(1)},\dots,Y^{(\nu)}) \in \mathbb{R}^\nu$ be a Gaussian vector with zero mean and non-degenerate covariance matrix $\Sigma = (\mathrm{E}Y^{(u)}Y^{(v)})_{1\le u,v\le\nu}$. Let $\mathbb{L}^2(\boldsymbol{Y})$ denote the class of all measurable functions $f : \mathbb{R}^\nu \to \mathbb{R}$ with $\mathrm{E}f^2(\boldsymbol{Y}) < \infty$. Let $m \ge 0$ be an integer. We say that $f \in \mathbb{L}^2(\boldsymbol{Y})$ *has a generalized Hermite rank not less than* m if either $m = 0$, or $m \ge 1$ and

$$\mathrm{E}[P(\boldsymbol{Y})f(\boldsymbol{Y})] = 0 \qquad \text{for all } P \in \mathcal{P}_{m-1}(\mathbb{R}^\nu) \tag{2.17}$$

hold, where $\mathcal{P}_m(\mathbb{R}^\nu)$ stands for the class of all polynomials P in variables $y^{(1)},\dots,y^{(\nu)}$ of degree m, that is, $P(\boldsymbol{y}) = \sum_{0\le|\boldsymbol{j}|\le m} c(\boldsymbol{j})\boldsymbol{y}^{\boldsymbol{j}} = \sum_{j^{(1)}\ge0,\dots,j^{(\nu)}\ge0:j^{(1)}+\dots+j^{(\nu)}\le m} c(j^{(1)},\dots,j^{(\nu)})\ (y^{(1)})^{j^{(1)}}\dots(y^{(\nu)})^{j^{(\nu)}}$.

Let $\boldsymbol{X} := \Sigma^{-1/2}\boldsymbol{Y}$, $\tilde f(\boldsymbol{x}) := f(\Sigma^{1/2}\boldsymbol{x})$. Then $\boldsymbol{X}$ has a standard Gaussian distribution in $\mathbb{R}^\nu$ and $\tilde f \in \mathbb{L}^2(\boldsymbol{X})$ with

$$\|\tilde f\|^2 = \mathrm{E}|\tilde f(\boldsymbol{X})|^2 = \mathrm{E}|f(\boldsymbol{Y})|^2. \tag{2.18}$$

The following proposition is known, see Nourdin *et al.* (2011, Proposition 2.1), Soulier (2001, p.195), but we include a proof of it for completeness.

Proposition 1 *Let $\boldsymbol{Y},\boldsymbol{X}$, $f\in\mathbb{L}^2(\boldsymbol{Y})$, $\tilde f\in\mathbb{L}^2(\boldsymbol{X})$ be defined as above and $m\ge0$ be a given integer. f has a generalized Hermite rank not less than m if and only if $\tilde f$ has a Hermite rank not less than m.*

Proof. The above proposition is true if $\boldsymbol{Y} = \boldsymbol{X}$ has a standard Gaussian distribution; see Soulier (2001, p.194). By definition

$$\mathrm{E}[P(\boldsymbol{Y})f(\boldsymbol{Y})] = \mathrm{E}[\tilde P(\boldsymbol{X})\tilde f(\boldsymbol{X})], \tag{2.19}$$

where $\tilde P(\boldsymbol{x}) := P(\Sigma^{1/2}\boldsymbol{x})$. Clearly, $P \in \mathcal{P}_{m-1}(\mathbb{R}^\nu)$ implies that $\tilde P \in \mathcal{P}_{m-1}(\mathbb{R}^\nu)$ is a polynomial of degree $m-1$. Therefore $\tilde f$ having a Hermite rank not less than m implies by (2.19) that f has a generalized Hermite rank not less than m. The converse statement again follows from (2.19), by taking $P(\boldsymbol{y}) = \hat P(\Sigma^{-1/2}\boldsymbol{y})$, where $\hat P \in \mathcal{P}_{m-1}(\mathbb{R}^\nu)$ is an arbitrary polynomial of degree $m-1$. □

Let $(\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_n)$ be a collection of Gaussian vectors $\boldsymbol{Y}_t = (Y_t^{(1)},\dots,Y_t^{(\nu)}) \in \mathbb{R}^\nu$ with zero mean $\mathrm{E}\boldsymbol{Y}_t = 0$ and non-degenerated covariance matrices $\Sigma_t = \big(\mathrm{Cov}\big(Y_t^{(u)},Y_t^{(v)}\big)\big)_{1\le u,v\le\nu}$, having a joint Gaussian distribution in $\mathbb{R}^{\nu n}$. Let $\varepsilon\in[0,1]$ be a fixed number. Call $(\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_n)$ $\varepsilon-$*correlated* if $\big|\mathrm{Cor}\big(Y_t^{(u)},Y_s^{(v)}\big)\big| \le \varepsilon$ for any $t\ne s$, $1\le t,s\le n$ and any $1\le u,v\le\nu$. Clearly, if the $\boldsymbol{Y}_t$'s are standard, this is equivalent to $(\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_n)$ being $\varepsilon-$standard.

We also use some elementary facts about matrix norms. Let $|\boldsymbol{x}| = \big(\sum_{i=1}^{\nu}(x^{(i)})^2\big)^{1/2}$ denote the Euclidean norm in $\mathbb{R}^\nu$, $A = (a_{ij})$ a real $\nu \times \nu-$matrix, $A^\intercal$ the transposed matrix, I the unit matrix, and $\|A\| := \sup_{|\boldsymbol{x}|=1}|A\boldsymbol{x}|$ the matrix spectral norm, respectively. Then $\|A\|_\infty := \max_{1\le i,j\le\nu}|a_{ij}| \le \|A\| \le \nu\|A\|_\infty$ and $\|AB\| \le \|A\|\,\|B\|$ for any such matrices A, B. An orthogonal matrix $O = (o_{ij})$ satisfies $OO^\intercal = O^\intercal O = I$ and $\|O\| = \|O^\intercal\| = 1$. Any symmetric matrix A can be written as $A = O^\intercal\Lambda O$, where O is an orthogonal matrix and Λ is a diagonal matrix. In addition, if A is positive definite, then $\|A\| = \|\Lambda\| = \lambda_{\max}$, $\|A^{-1}\| = \|\Lambda^{-1}\| = \lambda_{\min}^{-1}$, where $\lambda_{\max} \ge \lambda_{\min} > 0$ are the largest and the smallest eigenvalues of A. We shall also use the facts that for any symmetric positive definite matrix A,

$$\|A^{1/2}\| = \|A\|^{1/2}, \qquad \|A^{-1/2}\| = \|A^{-1}\|^{1/2}, \tag{2.20}$$

since $\|A^{1/2}\| = \|O^\intercal\Lambda^{1/2}O\| = \|\Lambda^{1/2}\| = \lambda_{\max}^{1/2}$, $\|A^{-1/2}\| = \|O^\intercal\Lambda^{-1/2}O\| = \|\Lambda^{-1/2}\| = \lambda_{\min}^{-1/2}$.

Corollary 1 *Let* $(\boldsymbol{Y}_1,\dots,\boldsymbol{Y}_n)$ *be an* $\varepsilon-$*correlated Gaussian vector,* $\boldsymbol{Y}_t = (Y_t^{(1)},\dots,Y_t^{(\nu)}) \in \mathbb{R}^\nu$ $(\nu \ge 1)$, *with zero mean* $\mathrm{E}\boldsymbol{Y}_t = 0$ *and non-degenerated covariance matrices* Σ_t *satisfying*

$$\max_{1\le t\le n}\|\Sigma_t^{-1}\| \ \le\ c_{\max} \tag{2.21}$$

for some constant $c_{\max} > 0$. *Let* $f_{j,t,n} \in \mathbb{L}^2(\boldsymbol{Y}_t), 1 \le j \le p\ (p \ge 2), 1 \le t \le n$ *be some functions. For given integers* $m \ge 1, 0 \le \alpha \le p, n \ge 1$, *let* Q_n *denote the sum in (2.2) where* $X_t^{(u)}$, $X_s^{(v)}$ *are replaced by* $Y_t^{(u)}, Y_s^{(v)}$, *respectively. Assume that the functions* $f_{1,t,n},\dots,f_{\alpha,t,n}$ *have a generalized Hermite rank at least equal to* m *for any* $n \ge 1, 1 \le t \le n$, *and that*

$$\varepsilon < \frac{1}{(\nu p - 1)\nu^2 c_{\max}}. \tag{2.22}$$

Then

$$\sum\nolimits' \big|\mathrm{E}\big[f_{1,t_1,n}(\boldsymbol{Y}_{t_1})\cdots f_{p,t_p,n}(\boldsymbol{Y}_{t_p})\big]\big| \ \ \le\ \ CKn^{p-\frac{\alpha}{2}}Q_n^{\frac{\alpha}{2}},$$

where $K := \prod_{j=1}^p \max_{1\le t\le n}\mathrm{E}^{1/2}\big[f_{j,t,n}^2(\boldsymbol{Y}_t)\big]$ *and the constant* $C = C(\varepsilon,p,m,\alpha,\nu,c_{\max})$ *depends on* $\varepsilon,p,m,\alpha,$ $\nu, c_{\max}$ *only.*

Proof. We will reduce the above inequality to that of Lemma 1, as follows. Let $\boldsymbol{X}_t := \Sigma_t^{-1/2}\boldsymbol{Y}_t$, $\tilde f_{j,t,n}(\boldsymbol{x}) := f_{j,t,n}(\Sigma_t^{1/2}\boldsymbol{x})$. The $\boldsymbol{X}_t$'s have a standard Gaussian distribution in $\mathbb{R}^\nu$ and the $\tilde f_{j,t,n}$'s satisfy $\|\tilde f_{j,t,n}\|^2 = \mathrm{E}\big[f_{j,t,n}^2(\boldsymbol{Y}_t)\big]$, see (2.18). By Proposition 1, $\tilde f_{j,t,n}, j = 1,\dots,\alpha$ have a Hermite rank not less than m. Next, using (2.20), (2.21) and the fact that the $\boldsymbol{Y}_t$'s are $\varepsilon-$correlated, for any $t \ne s$, $1 \le t,s \le n, 1 \le u,v \le \nu$

$$|\mathrm{E}X_t^{(u)}X_s^{(v)}| \ \le\ \nu^2\|\Sigma_t^{-1/2}\|_\infty\|\Sigma_s^{-1/2}\|_\infty \max_{1\le u,v\le\nu}|\mathrm{E}Y_t^{(u)}Y_s^{(v)}| \ \le\ \varepsilon\nu^2\|\Sigma_t^{-1/2}\|\|\Sigma_s^{-1/2}\| \ \le\ \varepsilon\nu^2 c_{\max} \tag{2.23}$$

This implies that the Gaussian vector $(\boldsymbol{X}_1,\dots,\boldsymbol{X}_n) \in \mathbb{R}^{\nu n}$ is $\tilde\varepsilon-$standard, where $\tilde\varepsilon := \varepsilon\nu^2 c_{\max}$. Then, in view of (2.22), (2.4) of Lemma 1 applies, according to which

$$\begin{aligned}&\sum\nolimits' \big|\mathrm{E}\big[f_{1,t_1,n}(\boldsymbol{Y}_{t_1})\cdots f_{p,t_p,n}(\boldsymbol{Y}_{t_p})\big]\big| \ =\ \sum\nolimits' \big|\mathrm{E}\big[\tilde f_{1,t_1,n}(\boldsymbol{X}_{t_1})\cdots \tilde f_{p,t_p,n}(\boldsymbol{X}_{t_p})\big]\big| \\ &\le\ C(\tilde\varepsilon,p,m,\alpha,\nu)\tilde K n^{p-\frac{\alpha}{2}}\tilde Q_n^{\frac{\alpha}{2}} \ \le\ C(\varepsilon,p,m,\alpha,\nu,c_{\max})Kn^{p-\frac{\alpha}{2}}Q_n^{\frac{\alpha}{2}},\end{aligned}$$

where $\tilde K, \tilde Q_n$ are the corresponding quantities in Lemma 1 (2.4) satisfying $\tilde K = K, \tilde Q_n \le (\varepsilon\nu^2 c_{\max})^m Q_n$ by (2.18), (2.23), respectively. □

We remark that condition (2.22) is not optimal since it does not reduce to (2.3) in the $\varepsilon-$standard case. This loss of optimality is due to the use of robust inequalities for matrix norms in (2.23).

Remark 1 As mentioned in the Introduction, Lemma 1 and Corollary 1 can be used for proving the tightness and the strong law of large numbers of various non-linear statistics from Gaussian observations. See Bardet and Surgailis (2011, 2012) on application for roughness estimation and Csörgő and Mielnichuk (1996), Koul and Surgailis (2002) for empirical process. The above-mentioned applications concern the 4th moment bound $M_n := \mathrm{E}\big(\sum_{t=1}^n f_{t,n}(\boldsymbol{Y}_n(t))\big)^4 = O(n^{-\kappa})$ for a suitable $\kappa > 0$, where $(\boldsymbol{Y}_n(t)), (f_{t,n})$ satisfy similar conditions as in Corollary 1. Clearly, $M_n = \sum_{t_1,\ldots,t_4=1}^n \mathrm{E}\big[\prod_{i=1}^4 f_{t_i,n}(\boldsymbol{Y}_n(t_i))\big]$ can be decomposed into four terms according to the number of coinciding "diagonals" $t_i = t_j$ in the last sum, where each term can be estimated with the help of Corollary 1. Let us note that condition (2.22) in the above applications is guaranteed by a preliminary "decimation" of the sum $\sum_{t=1}^n f_{t,n}(\boldsymbol{Y}_n(t)$, see (Csörgő and Mielnichuk, 1996) and (Bardet and Surgailis, 2012) for details.

3 A CLT for triangular array of functions of Gaussian vectors

Let $(\boldsymbol{X}_n(k))_{1\le k\le n, n\in\mathbb{N}}$ be a triangular array of standardized Gaussian vectors with values in $\mathbb{R}^\nu$, $\boldsymbol{X}_n(k) = (X_n^{(1)}(k),\ldots,X_n^{(\nu)}(k))$, $\mathrm{E}X_n^{(p)}(k) = 0$, $\mathrm{E}X_n^{(p)}(k)X_n^{(q)}(k) = \delta_{pq}$. Now define,

$$r_n^{(p,q)}(j,k) := \mathrm{E}X_n^{(p)}(j)X_n^{(q)}(k) \qquad (1 \le j,k \le n).$$

For a given integer $m \ge 1$, introduce the following assumptions: for any $1 \le p,q \le \nu$,

$$\sup_{n\ge1} \max_{1\le k\le n} \sum_{1\le j\le n} \big|r_n^{(p,q)}(j,k)\big|^m \;<\; \infty, \tag{3.1}$$

$$\sup_{n\ge1} \frac{1}{n} \sum_{\substack{1\,\le\, j,\,k\,\le\, n\\ |j\,-\,k|\,>\,K}} \Big|r_n^{(p,q)}(j,k)\Big|^m \underset{K\to\infty}{\longrightarrow} 0, \tag{3.2}$$

$$\forall (j,k)\in\{1,\ldots,n\}^2, \qquad \Big|r_n^{(p,q)}(j,k)\Big| \;\le\; |\rho(j-k)| \quad \text{with} \;\sum_{j\in\mathbb{Z}} |\rho(j)|^m < \infty. \tag{3.3}$$

Note (3.3) $\Rightarrow$ (3.1) and (3.3) $\Rightarrow$ (3.2). Let $\mathbb{L}_0^2(\boldsymbol{X}) := \{f \in \mathbb{L}^2(\boldsymbol{X}) : \mathrm{E}f(\boldsymbol{X}) = 0\}$, where $\boldsymbol{X} \in \mathbb{R}^\nu$ denotes a standard Gaussian vector as above.

Theorem 1 *Let $(\boldsymbol{X}_n(k))_{1\le k\le n, n\in\mathbb{N}}$ be a triangular array of standardized Gaussian vectors.*

(i) Assume (3.1). Let $f_k \in \mathbb{L}_0^2(\boldsymbol{X})$ $(1 \le k \le n)$ have a Hermite rank at least $m \in \mathbb{N}^*$. Then there exists a constant C independent of n and $f_k, 1 \le k \le n$ such that

$$\mathrm{E}\Big(n^{-1/2}\sum_{k=1}^n f_k\left(\boldsymbol{X}_n(k)\right)\Big)^2 \;\le\; C \max_{1\le k\le n} \|f_k\|^2. \tag{3.4}$$

(ii) Assume (3.1) and (3.2). Let $f_{k,n} \in \mathbb{L}_0^2(\boldsymbol{X})$ $(n \ge 1, 1 \le k \le n)$ be a triangular array of functions all having Hermite rank at least $m \in \mathbb{N}^*$. Assume that there exists a $\mathbb{L}_0^2(\boldsymbol{X})-$valued continuous function $\phi_\tau, \tau \in [0,1]$, such that

$$\sup_{\tau\in(0,1]} \|f_{[\tau n],n} - \phi_\tau\|^2 = \sup_{\tau\in(0,1]} \mathrm{E}\big(f_{[\tau n],n}(\boldsymbol{X}) - \phi_\tau(\boldsymbol{X})\big)^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{3.5}$$

Moreover, let

$$\sigma_n^2 := \mathrm{E}\Big(n^{-1/2}\sum_{k=1}^n f_{k,n}\left(\boldsymbol{X}_n(k)\right)\Big)^2 \underset{n\to\infty}{\longrightarrow} \sigma^2, \tag{3.6}$$

where $\sigma^2 > 0$. Then

$$n^{-1/2}\sum_{k=1}^n f_{k,n}\left(\boldsymbol{X}_n(k)\right) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \mathcal{N}(0,\sigma^2). \tag{3.7}$$

(iii) Assume (3.3). Moreover, assume that for any $\tau \in [0,1]$ and any $J \in \mathbb{N}^*$,

$$\big(\boldsymbol{X}_n([n\tau]+j)\big)_{-J\le j\le J} \ \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \ \big(\boldsymbol{W}_\tau(j)\big)_{-J\le j\le J}, \tag{3.8}$$

where $(\boldsymbol{W}_\tau(j))_{j\in\mathbb{Z}}$ is a stationary Gaussian process taking values in $\mathbb{R}^\nu$ and depending on parameter $\tau \in (0,1)$. Let $f_{k,n} \in \mathbb{L}^2_0(\boldsymbol{X})$ ($n \ge 1, 1 \le k \le n$) satisfy the same conditions as in part (ii), with exception of (3.6). Then (3.6) and (3.7) hold, with

$$\sigma^2 \ = \ \int_0^1 \mathrm{d}\tau \Big(\sum_{j\in\mathbb{Z}} \mathrm{E}\left[\phi_\tau\left(\boldsymbol{W}_\tau(0)\right)\phi_\tau\left(\boldsymbol{W}_\tau(j)\right)\right] \Big). \tag{3.9}$$

We remark that parts (i) and (ii) of Theorem 1 are natural extensions of Theorem 2 of Arcones (1994) (for instance, condition (3.1) is the same as condition (2.40) of Arcones (1994) in the case of stationary sequences). We expect that parts (i) and (ii) can be also obtained following the method in Nourdin *et al.* (2010). Part (iii) seems more interesting. In Bardet and Surgailis (2011), (iii) is applied when $\boldsymbol{X}_n(j) = \boldsymbol{Z}_{j/n}$ and $(\boldsymbol{Z}_t)_t$ is a vector valued continuous time process.

Similarly to Lemma 1, Theorem 1 can be extended to nonstandardized Gaussian vectors. Corollary 2 below refers to the most interesting part (iii) of Theorem 1.

Corollary 2 *Let $\boldsymbol{Y}_n(k) = \big(Y_n^{(1)}(k),\ldots,Y_n^{(\nu)}(k)\big) \in \mathbb{R}^\nu$, $1 \le k \le n$, $n \in \mathbb{N}$ be a triangular array of jointly Gaussian vectors, with zero mean $\mathrm{E}\boldsymbol{Y}_n(k) = 0$ and non-degenerate covariance matrices $\Sigma_{k,n} = \mathrm{E}\boldsymbol{Y}_n(k)\boldsymbol{Y}_n(k)^\intercal$. Assume that covariances $r_n^{(p,q)}(j,k) := \mathrm{Cov}\big(Y_n^{(p)}(j),Y_n^{(q)}(k)\big)$ satisfy (3.3), for some $m \ge 1$. Moreover, assume that (3.8) holds with $\boldsymbol{X}_n(\cdot)$ replaced by $\boldsymbol{Y}_n(\cdot)$, where $(\boldsymbol{W}_\tau(j))_{j\in\mathbb{Z}}$ is a stationary Gaussian $\mathbb{R}^\nu$-valued process with non-degenerate covariance matrix $\Sigma_\tau := \mathrm{E}\boldsymbol{W}_\tau(0)\boldsymbol{W}_\tau(0)^\intercal$ such that*

$$\sup_{\tau\in(0,1]} \|\Sigma_\tau^{-1}\| < \infty \tag{3.10}$$

and

$$\sup_{\tau\in(0,1]} \|\Sigma_{[n\tau],n} - \Sigma_\tau\| \ \underset{n\to\infty}{\longrightarrow} \ 0. \tag{3.11}$$

Let $f_{k,n} \in \mathbb{L}^2_0(\boldsymbol{Y}_n(k))$, $1 \le k \le n$, $n \in \mathbb{N}$ be a triangular array of functions all having a generalized Hermite rank not less than m and such that

$$\sup_{\tau\in(0,1]} \mathrm{E}\left(\tilde{f}_{[\tau n],n}(\boldsymbol{X}) - \tilde{\phi}_\tau(\boldsymbol{X})\right)^2 \ \underset{n\to\infty}{\longrightarrow} \ 0, \tag{3.12}$$

where $\tilde{f}_{k,n}(\boldsymbol{x}) := f_{k,n}(\Sigma_{k,n}^{1/2}\boldsymbol{x})$ and where $\tilde{\phi}_\tau, \tau \in [0,1]$ is a $\mathbb{L}^2_0(\boldsymbol{X})-$valued continuous function, with $\boldsymbol{X}$ a standard Gaussian vector in $\mathbb{R}^\nu$ as usual. Then

$$n^{-1/2}\sum_{k=1}^n f_{k,n}\big(\boldsymbol{Y}_n(k)\big) \ \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \ \mathcal{N}(0,\sigma^2). \tag{3.13}$$

where σ^2 is defined in (3.9), with $\phi_\tau(\boldsymbol{x}) := \tilde{\phi}_\tau(\Sigma_\tau^{-1/2}\boldsymbol{x})$.

Proof of Corollary 2. Similarly as in the proof of Corollary 1, let $\boldsymbol{X}_n(k) := \Sigma_{k,n}^{-1/2}\boldsymbol{Y}_n(k)$. The $\boldsymbol{X}_n(k)$'s are standardized Gaussian vectors in $\mathbb{R}^\nu$ and the $\tilde{f}_{k,n}$'s belong to $\mathbb{L}^2(\boldsymbol{X})$ and have a Hermite rank not less than m. Assumptions (3.8) and (3.11) entail for any $\tau \in (0,1)$, $\big(\boldsymbol{X}_n(j+[n\tau]))\big)_{-J\le j\le J} \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \big(\widetilde{\boldsymbol{W}}_\tau(j)\big)_{-J\le j\le J}$, where $\widetilde{\boldsymbol{W}}_\tau(j) := \Sigma_\tau^{-1/2}\boldsymbol{W}_\tau(j)$, $j \in \mathbb{Z}$ is a stationary Gaussian process having a unit covariance matrix $\mathrm{E}\widetilde{\boldsymbol{W}}_\tau(0)\widetilde{\boldsymbol{W}}_\tau(0)^\intercal = I$. Conditions (3.10) and (3.11) imply that $\max_{1\le k\le n}\|\Sigma_{k,n}^{-1/2}\| \le C$. The last fact together with condition (3.3) for covariances $r_n^{(p,q)}(j,k) := \mathrm{Cov}\big(Y_n^{(p)}(j),Y_n^{(q)}(k)\big)$ imply a similar condition for

$\mathrm{Cov}\big(X_n^{(p)}(j), X_n^{(q)}(k)\big)$: for all $(j,k) \in \{1,\cdots,n\}^2$ we have that $\max_{1\le u,v\le \nu} \big|\mathrm{E}X_n^{(u)}(j)X_n^{(v)}(k)\big| \le C|\rho(j-k)|$; see (2.23). This way we see that the conditions of Theorem 1(iii) including (3.5) are satisfied and can be applied to the families of Gaussian vectors $(\boldsymbol{X}_n(k))$ and functions $\big(\tilde f_{k,n}\big)$, yielding (3.13). $\square$

Proof of Theorem 1. (i) Using Arcones' inequality (see Arcones, 1994, (2.44) or Soulier, 2001, (2.4)), one obtains

$$\begin{aligned}\mathrm{E}\Big(n^{-1/2}\sum_{k=1}^n f_k\left(\boldsymbol{X}_n(k)\right)\Big)^2 &= \frac{1}{n}\sum_{k=1}^n \|f_k\|^2 + \frac{1}{n}\sum{}' \,\mathrm{E}f_k(\boldsymbol{X}_n(k))f_\ell(\boldsymbol{X}_n(\ell)) \\ &\le \max_{1\le k\le n}\|f_k\|^2 + C\big(\max_{1\le k\le n}\|f_k\|\big)^2 \max_{1\le k\le n}\sum_{1\le \ell\le n, \ell\ne k}\max_{1\le p,q\le\nu}\big|r_n^{(p,q)}(k,\ell)\big|^m,\end{aligned}$$

where C is a positive real number not depending on n or f_k. Now, using assumption (3.1), (i) is proved.

(ii) We use the following well-known fact. Let $(Z_n)_{n\ge1}$ be a sequence of r.v.'s with zero mean and finite variance. Then $Z_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} \mathcal{N}(0,\sigma^2)$ if and only if for any $\epsilon>0$ one can find an integer $n_0(\epsilon)\ge1$ and a sequence $(Z_{n,\epsilon})_{n\ge1}$ satisfying $Z_{n,\epsilon} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} \mathcal{N}(0,\sigma_\epsilon^2)$ and $\forall n>n_0(\epsilon)$, $\mathrm{E}(Z_n - Z_{n,\epsilon})^2 < \epsilon$.

Let $Z_n := n^{-1/2}\sum_{k=1}^n f_{k,n}\left(\boldsymbol{X}_n(k)\right)$. We shall construct an approximating sequence $Z_{n,\epsilon}$ with the above properties in two steps.

Firstly, by condition (3.5) and continuity of ϕ_τ, for a given $\epsilon>0$ one can find integers $M, n_0(\epsilon)$ and a partition $0 =: \tau_0 < \tau_1 < \ldots < \tau_M < \tau_{M+1} := 1$ such that $\forall\ n > n_0(\epsilon)$,

$$\max_{0\le i\le M}\max_{k/n\in(\tau_i,\tau_{i+1}]}\|f_{k,n}-\phi_{\tau_i}\| = \max_{0\le i\le M}\max_{k/n\in(\tau_i,\tau_{i+1}]}\big(\mathrm{E}(f_{k,n}(\boldsymbol{X})-\phi_{\tau_i}(\boldsymbol{X}))^2\big)^{1/2} \ < \ \epsilon. \tag{3.14}$$

Put

$$\widetilde{Z}_{n,\epsilon} := n^{-1/2}\sum_{i=0}^M \sum_{k/n\in(\tau_i,\tau_{i+1}]}\phi_{\tau_i}\left(\boldsymbol{X}_n(k)\right).$$

Note for any $\tau\in(0,1]$, the function ψ_τ has Hermite rank not less than m, being the limit of a sequence of $\mathbb{L}_0^2(\boldsymbol{X})-$valued functions of Hermite rank $\ge m$. Therefore for the difference $Z_n - \widetilde{Z}_{n,\epsilon}$ the inequality (3.4) applies, yielding $\forall\ n>n_0(\epsilon)$

$$\mathrm{E}(Z_n-\widetilde{Z}_{n,\epsilon})^2 \ \le\ C\max_{0\le i\le M}\max_{k/n\in(\tau_i,\tau_{i+1}]}\|f_{k,n}-\phi_{\tau_i}\|^2 \le C\epsilon^2 \tag{3.15}$$

in view of (3.14), with a constant C independent of n,ϵ.

Secondly, we expand each ϕ_{τ_i} in Hermite polynomials:

$$\phi_{\tau_i}(\boldsymbol{x}) \ = \ \sum_{m\le|\boldsymbol{k}|}\frac{J_i(\boldsymbol{k})}{\boldsymbol{k}!}H_{\boldsymbol{k}}(\boldsymbol{x}), \qquad (i=0,1,\ldots,M) \tag{3.16}$$

where

$$J_i(\boldsymbol{k}) := J_{\phi_{\tau_i}}(\boldsymbol{k}) = \mathrm{E}\phi_{\tau_i}(\boldsymbol{X})H_{\boldsymbol{k}}(\boldsymbol{X}), \qquad |J_i(\boldsymbol{k})| \le \|\phi_{\tau_i}\|(\boldsymbol{k}!)^{1/2}.$$

We can choose $t(\epsilon)\in\mathbb{N}$ large enough so that

$$\|\phi_{\tau_i}-\phi_{\tau_i,\epsilon}\| \le \epsilon, \qquad (i=0,1,\ldots,M), \tag{3.17}$$

where $\phi_{\tau_i,\epsilon}$ is a finite sum of Hermite polynomials:

$$\phi_{\tau_i,\epsilon}(\boldsymbol{x}) \ := \ \sum_{m\le|\boldsymbol{k}|\le t(\epsilon)}\frac{J_i(\boldsymbol{k})}{\boldsymbol{k}!}H_{\boldsymbol{k}}(\boldsymbol{x}), \qquad (i=0,1,\ldots,M). \tag{3.18}$$

Note $t(\epsilon)$ does not depend on $i = 0, 1, \ldots, M$, and $\epsilon > 0$ is the same as in (3.14). Put

$$Z_{n,\epsilon} := n^{-1/2} \sum_{i=0}^{M} \sum_{k/n \in (\tau_i, \tau_{i+1}]} \phi_{\tau_i,\epsilon}\left(\boldsymbol{X}_n(k)\right). \tag{3.19}$$

Applying (3.4) to the difference $\widetilde{Z}_{n,\epsilon} - Z_{n,\epsilon}$ and using (3.17) and (3.15), we obtain $\forall\ n > n_0(\epsilon)$,

$$\mathrm{E}(Z_n - Z_{n,\epsilon})^2 \ \le\ C\epsilon^2 \tag{3.20}$$

where the constant C is independent of n, ϵ. Let $\sigma^2_{n,\epsilon} := \mathrm{E}Z^2_{n,\epsilon}$. From (3.20) and condition (3.6) it follows that $\forall n > n_0(\epsilon)$,

$$\sigma^2 - C\epsilon \ \le\ \sigma^2_{n,\epsilon} \ \le\ \sigma^2 + C\epsilon, \tag{3.21}$$

with some C independent of n, ϵ. In particular, by choosing $\epsilon > 0$ small enough, it follows that $\liminf_{n\to\infty} \sigma^2_{n,\epsilon} > 0$. We shall prove below that for any fixed $\epsilon > 0$,

$$U_n \ :=\ \frac{Z_{n,\epsilon}}{\sigma_{n,\epsilon}} \ =\ \frac{1}{\sigma_{n,\epsilon} n^{1/2}} \sum_{i=1}^{M} \sum_{k/n \in (\tau_i, \tau_{i+1}]} \phi_{\tau_i,\epsilon}\left(\boldsymbol{X}_n(k)\right) \ \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}\ \mathcal{N}(0,1). \tag{3.22}$$

As noted in the beginning of the proof of the theorem, the CLT in (3.7) follows from (3.22), (3.20), (3.21). Indeed, write

$$\begin{aligned}
\mathrm{E}\mathrm{e}^{\mathrm{i}aZ_n} - \mathrm{e}^{-a^2\sigma^2/2} \ &=\ \left(\mathrm{E}\mathrm{e}^{\mathrm{i}aZ_n} - \mathrm{E}\mathrm{e}^{\mathrm{i}aZ_{n,\epsilon}}\right) + \left(\mathrm{E}\mathrm{e}^{\mathrm{i}a\sigma_{n,\epsilon}U_n} - \mathrm{e}^{-a^2\sigma^2_{n,\epsilon}/2}\right) \\
&+\ \left(\mathrm{e}^{-a^2\sigma^2_{n,\epsilon}/2} - \mathrm{e}^{-a^2\sigma^2/2}\right) \ :=\ \sum_{i=1}^{3} \ell_i(n).
\end{aligned}$$

Here, for some constant C independent of n, a, ϵ,

$$\begin{aligned}
|\ell_1(n)| \ &\le\ \mathrm{E}^{1/2}\left|\mathrm{e}^{\mathrm{i}a(Z_n - Z_{n,\epsilon})} - 1\right|^2 \ \le\ |a|\mathrm{E}^{1/2}|Z_n - Z_{n,\epsilon}|^2 \ \le\ C|a|\epsilon, \\
|\ell_3(n)| \ &\le\ Ca^2\left|\sigma^2_{n,\epsilon} - \sigma^2\right| \ \le\ Ca^2\epsilon,
\end{aligned}$$

and therefore $\ell_i(n), i = 1, 3$ can be made arbitrarily small by choosing $\epsilon > 0$ small enough; see (3.20), (3.21). On the other hand, the convergence in (3.22) implies uniform convergence of characteristic functions on compact intervals and therefore $\sup_{|a|\le A} |\ell_2(n)| \le \sup_{|a|\le 2A} \left|\mathrm{E}\mathrm{e}^{\mathrm{i}aU_n} - \mathrm{e}^{-a^2/2}\right| \underset{n\to\infty}{\longrightarrow} 0$ for any $A > 0$. This proves (3.7).

It remains to prove (3.22). The proof of the corresponding CLTs for sums of Hermite polynomials in Arcones (1994) and Breuer and Major (1983) refer to stationary processes and use Fourier methods. Therefore we present an independent proof of (3.22) based on cumulants and the Hölder inequality in (2.14). Again, our proof appears to be much simpler than computations in the above mentioned papers.

Accordingly, it suffices to show that cumulants of order $p \ge 3$ of U_n asymptotically vanish. In view of (3.21) and linearity of cumulants, this follows from the fact that for any $p \ge 3$ and any multiindices $\boldsymbol{k}_u = (k_u^{(1)}, \ldots, k_u^{(\nu)}) \in \mathbb{Z}_+^\nu,\ u = 1, \ldots, p$ with $k_u = |\boldsymbol{k}_u| = k_u^{(1)} + \ldots + k_u^{(\nu)} \ge m$ $(1 \le u \le p)$,

$$\Sigma_n \ :=\ \sum_{t_1,\ldots,t_p=1}^{n} |\mathrm{cum}(t_1, \ldots, t_p)| = \ o(n^{p/2}), \tag{3.23}$$

where $\mathrm{cum}(t_1, \ldots, t_p)$ stands for joint cumulant:

$$\mathrm{cum}(t_1, \ldots, t_p) \ :=\ \mathrm{cum}\left(H_{\boldsymbol{k}_1}(\boldsymbol{X}_n(t_1)), \ldots, H_{\boldsymbol{k}_p}(\boldsymbol{X}_n(t_p))\right). \tag{3.24}$$

Split $\Sigma_n = \Sigma_n'(K) + \Sigma_n''(K)$, where

$$\Sigma_n'(K) \ :=\ \sum_{t_1,\ldots,t_p=1}^{n} |\mathrm{cum}(t_1, \ldots, t_p)|\, \mathbf{1}(|t_i - t_j| \le K\ \forall i \ne j)$$

and where K will be chosen large enough. Then for any fixed K, we have $\Sigma'_n(K) = O(n) = o(n^{p/2})$ as $p \ge 3$. The remaining sum $\Sigma''_n(K)$ does not exceed $\sum_{1\le i\ne j\le p} \Sigma''_{n,i,j}(K)$, where

$$\Sigma''_{n,i,j}(K) \ := \sum_{t_1,\dots,t_p=1}^{n} |\mathrm{cum}(t_1,\dots,t_p)|\,\mathbf{1}(|t_i - t_j| > K).$$

Therefore, relation (3.23) follows if we show that there exist $\delta(K) \underset{K\to\infty}{\longrightarrow} 0$ and $\tilde n_0$ such that for any $1 \le i \ne j \le p$ and any $n > \tilde n_0$

$$\limsup_{n\to\infty} \Sigma''_{n,i,j}(K) \ < \ \delta(K) n^{p/2}. \tag{3.25}$$

The proof below is limited to $(i,j) = (1,2)$ as the general case is analogous. It is well-known that the joint cumulant in (3.24), similarly to the joint moment in (2.6), can be expressed as a sum over all *connected* diagrams $\gamma \in \Gamma_c(T)$ over the table T in (2.7). By introducing $\bar\rho(s,t) := \max_{1\le p,q\le\nu} \left| r_n^{(p,q)}(s,t)\right|$, we obtain

$$|\mathrm{cum}(t_1,\dots,t_p)| \ \le \sum_{\gamma\in\Gamma_c(T)} \prod_{1\le u<v\le p} (\bar\rho(t_u,t_v))^{\ell_{uv}}, \tag{3.26}$$

where we use the notation in (2.6). Therefore,

$$\Sigma''_{n,1,2}(K) \ \le \sum_{\gamma\in\Gamma_c(T)} \sum_{t_1,\dots,t_p=1}^{n} \prod_{1\le u<v\le p} (\bar\rho(t_u,t_v))^{\ell_{uv}} \mathbf{1}(|t_1-t_2|>K) \ := \sum_{\gamma\in\Gamma_c(T)} \bar I_{n,T}(\gamma),$$

Next, by applying the Hölder inequality as in (2.14),

$$\bar I_{n,T}(\gamma) \quad \le \quad \min\left(\prod_{1\le u<v\le p} \bar R_{uv}, \prod_{1\le u<v\le p} \bar R_{vu} \right). \tag{3.27}$$

where (cf. (2.12))

$$\bar R_{uv} \quad := \quad \begin{cases} \left(\sum_{1\le t\le n}\left(\sum_{1\le s\le n} \bar\rho^{k_u}(s,t)\right)^{k_v/k_u}\right)^{\ell_{uv}/k_v}, & (u,v)\ne(1,2),(2,1), \\ \left(\sum_{1\le t\le n}\left(\sum_{1\le s\le n} \bar\rho^{k_1}(s,t)\mathbf{1}(|t-s|>K)\right)^{k_2/k_1}\right)^{\ell_{12}/k_2}, & (u,v)=(1,2), \\ \left(\sum_{1\le t\le n}\left(\sum_{1\le s\le n} \bar\rho^{k_2}(t,s)\mathbf{1}(|t-s|>K)\right)^{k_1/k_2}\right)^{\ell_{12}/k_1}, & (u,v)=(2,1). \end{cases}$$

From assumptions (3.1), (3.2), there exists a constant C and $\delta(K) \underset{K\to\infty}{\longrightarrow} 0$ independent of n such that for any $k \ge m$ and any $n \ge 1$

$$\sup_{1\le t\le n} \sum_{s=1}^{n} \bar\rho^k(s,t) \quad \le \quad Cn,$$

$$\sup_{1\le t\le n} \sum_{s=1}^{n} \bar\rho^k(s,t)\mathbf{1}(|t-s|>K) \quad \le \quad \delta(K)n.$$

Therefore

$$\bar R_{uv} \ \le \ \begin{cases} Cn^{\ell_{uv}/k_v}, & (u,v)\ne(1,2),(2,1), \\ \tilde\delta(K)n^{\ell_{12}/k_2}, & (u,v)=(1,2), \\ \tilde\delta(K)n^{\ell_{12}/k_1}, & (u,v)=(2,1), \end{cases}$$

with some $\tilde\delta(K) \underset{K\to\infty}{\longrightarrow} 0$ independent of n. Consequently, the minimum on the right-hand side of (3.27) does not exceed

$$C\tilde\delta(K)\min\left(n^{\sum_{1\le u<v\le p}\ell_{uv}/k_v}, n^{\sum_{1\le u<v\le p}\ell_{uv}/k_u}\right) \ = \ C\tilde\delta(K)n^{\min(L(T),L^*(T))}$$

where the quantities $L(T), L^*(T)$ introduced in (2.15) satisfy $L(T) + L^*(T) = p$, see (2.16), and therefore $\min(L(T), L^*(T)) \le p/2$. This proves (3.25) and the CLT in (3.22), thereby completing the proof of part (ii).

(iii) Let us first prove (3.6) with σ^2 given in (3.9) in the case when $f_{k,n} \equiv f$ do not depend on k, n (in such case, one has $\phi_\tau \equiv f$, too). We have

$$\sigma_n^2 \quad = \quad n^{-1} \sum_{k,k'=1}^{n} \mathrm{E}\big[f\left(\boldsymbol{X}_n(k)\right) f\left(\boldsymbol{X}_n(k')\right) \big] \quad = \quad \int_0^1 F_n(\tau) \mathrm{d}\tau, \tag{3.28}$$

where

$$F_n(\tau) \ := \sum_{j=1-[n\tau]}^{n-[n\tau]} \mathrm{E}\big[f\left(\boldsymbol{X}_n([n\tau])\right) f\left(\boldsymbol{X}_n([n\tau]+j)\right) \big]. \tag{3.29}$$

Condition (3.8) implies that

$$\mathrm{E}\big[f\left(\boldsymbol{X}_n([n\tau])\right) f\left(\boldsymbol{X}_n([n\tau]+j)\right) \big] \to \mathrm{E}\big[f\left(\boldsymbol{W}_\tau(0)\right) f\left(\boldsymbol{W}_\tau(j)\right) \big]$$

for each $j \in \mathbb{Z}$ as $n \to \infty$. From (3.3) and with the inequality of previous part (i), it exists $C > 0$ such that

$$\Big| \mathrm{E}\big[f\left(\boldsymbol{X}_n([n\tau])\right) f\left(\boldsymbol{X}_n([n\tau]+j)\right) \big] \Big| \ \le \ C|\rho(j)|^m, \tag{3.30}$$

and $\sum_{j\in\mathbb{Z}} |\rho(j)|^m < \infty$. Hence, from Lebesgue Theorem,

$$F_n(\tau) = \sum_{j\in\mathbb{Z}} \mathbf{1}_{j\in\{1-[n\tau],\cdots,n-[n\tau]\}} \mathrm{E}\big[f\left(\boldsymbol{X}_n([n\tau])\right) f\left(\boldsymbol{X}_n([n\tau]+j)\right) \big] \underset{n\to\infty}{\longrightarrow} \sum_{j\in\mathbb{Z}} \mathrm{E}\big[f\left(\boldsymbol{W}_\tau(0)\right) f\left(\boldsymbol{W}_\tau(j)\right) \big].$$

The dominated convergence theorem allows one to pass to the limit under the integral, thereby proving (3.6) with σ^2 given in (3.9) in the case $f_{k,n} \equiv f$.

To end the proof, consider the general case of $f_{k,n}$ as in (iii). Let $Z_{n,\epsilon}$ be defined as in (3.19). Note relation (3.20) holds as its proof does not use (3.6). In part (ii), we used (3.6) to prove (3.21). Now we want to prove (3.21) using (3.8) instead of (3.6). This will suffice for the proof of (iii), as the remaining argument is the same as in part (ii).

Consider the variance $\sigma^2_{n,\epsilon} = \mathrm{E}Z^2_{n,\epsilon}$ of $Z_{n,\epsilon}$ defined in (3.19):

$$\sigma^2_{n,\epsilon} \quad = \quad n^{-1} \left(\sum_{0\le i\le M} \mathrm{E}D_i^2 + 2 \sum_{0\le i<j\le M} \mathrm{E}D_iD_j \right),$$

where

$$D_i \ := \sum_{k/n\in[\tau_i,\tau_{i+1})} \phi_{\tau_i,\epsilon}\left(\boldsymbol{X}_n(k)\right).$$

Let us show that for ϵ, M fixed, and as $n \to \infty$,

$$\mathrm{E}D_iD_j \quad = \quad o(n) \qquad (i \ne j), \tag{3.31}$$

$$n^{-1}\mathrm{E}D_i^2 \quad \underset{n\to\infty}{\longrightarrow} \quad \int_{\tau_i}^{\tau_{i+1}} \sum_{j\in\mathbb{Z}} \mathrm{E}\big[\phi_{\tau_i,\epsilon}\left(\boldsymbol{W}_\tau(0)\right) \phi_{\tau_i,\epsilon}\left(\boldsymbol{W}_\tau(j)\right) \big] \mathrm{d}\tau. \tag{3.32}$$

Here, (3.32) follows from the argument in the beginning of the proof of (iii), as $\phi_{\tau_i,\epsilon}$ does not depend on k, n. Relation (3.31) is implied by the following computations. Using the Hermitian rank of functions $\phi_{\tau_i,\epsilon}$, for $i < j$ one obtains

$$\begin{aligned}
\left| \mathrm{E}\phi_{\tau_i,\epsilon}\left(\boldsymbol{X}_n([n\tau_i]+k)\right) \phi_{\tau_j,\epsilon}\left(\boldsymbol{X}_n([n\tau_j]+\ell)\right) \right| \quad &\le \quad C\|\phi_{\tau_i,\epsilon}\| \cdot \|\phi_{\tau_j,\epsilon}\| \max_{1\le p,q\le\nu} \left| r_n^{(p,q)}([n\tau_i]+k, [n\tau_j]+\ell) \right|^m \\
&\le \quad C\|\phi_{\tau_i,\epsilon}\| \cdot \|\phi_{\tau_j,\epsilon}\| \big| \rho([n\tau_j]-[n\tau_i]+\ell-k) \big|^m.
\end{aligned}$$

Therefore, for $i < j$, and ε small enough,

$$\begin{aligned}
\big|\mathrm{E}D_iD_j\big| &\le C\max_{\tau\in[0,1]}\|\phi_\tau\|^2\sum_{k=0}^{[\tau_{i+1}n]-[\tau_i n]}\sum_{\ell=0}^{[\tau_{j+1}n]-[\tau_j n]}\big|\rho([n\tau_j]-[n\tau_i]+\ell-k)\big|^m\\
&\le C\max_{\tau\in[0,1]}\|\phi_\tau\|^2\sum_{k=1}^{n}k\big|\rho(k)\big|^m \;=\; o(n)
\end{aligned}$$

since $\sum_{k=1}^n k\big|\rho(k)\big|^m \le \sqrt{n}\sum_{1\le k\le\sqrt{n}}\big|\rho(k)\big|^m + n\sum_{k>\sqrt{n}}\big|\rho(k)\big|^m = o(n)$. Thus, (3.31) is proved. From (3.31), (3.32) it follows that for any $\epsilon > 0$

$$\lim_{n\to\infty}\sigma^2_{n,\epsilon} \;=\; \bar\sigma^2_\epsilon \;:=\; \sum_{i=0}^{M}\int_{\tau_i}^{\tau_{i+1}}\sum_{j\in\mathbb{Z}}\mathrm{E}\big[\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(0))\,\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(j))\big]\mathrm{d}\tau.$$

Consider the difference $\bar\sigma^2_\epsilon - \sigma^2 = \sum_{i=0}^M\int_{\tau_i}^{\tau_{i+1}}\sum_{j\in\mathbb{Z}}\Theta_{M,\epsilon}(\tau,j)\mathrm{d}\tau$, where

$$\begin{aligned}
|\Theta_{M,\epsilon}(\tau,j)| &= |\mathrm{E}\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(0))\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(j)) - \mathrm{E}\phi_\tau(\boldsymbol{W}_\tau(0))\phi_\tau(\boldsymbol{W}_\tau(j))|\\
&\le |\mathrm{E}\left(\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(0)) - \phi_\tau(\boldsymbol{W}_\tau(0))\right)\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(j))| \;+\; |\mathrm{E}\left(\phi_{\tau_i,\epsilon}(\boldsymbol{W}_\tau(j)) - \phi_\tau(\boldsymbol{W}_\tau(j))\right)\phi_\tau(\boldsymbol{W}_\tau(0))|\\
&\le \|\phi_{\tau_i,\epsilon} - \phi_\tau\|\left(\|\phi_{\tau_i,\epsilon}\| + \|\phi_\tau\|\right). \qquad (3.33)
\end{aligned}$$

Using uniform continuity of $\phi_\tau, \tau\in[0,1]$ (in the sense of $\mathbb{L}^2-$norm continuity), we obtain that the right-hand side of (3.33) can be made arbitrarily small by choosing M (= the number of partition intervals of $[0,1]$) and $t(\epsilon)$ (= the truncation level of Hermite expansion) sufficiently large, uniformly in $\tau\in[0,1]$ and $j\in\mathbb{Z}$. On the other hand, $|\Theta_{M,\epsilon}(\tau,j)| \le C\sup_{\tau\in[0,1]}\|\phi_\tau\|^2|\rho(j)|^m$ by Arcones' inequality, c.f. (3.30). Therefore $|\Theta_{M,\epsilon}(\tau,j)|$ is dominated by a summable function uniformly in M,ϵ. Now, (3.21) follows by an application of Lebesgue theorem. This proves part (iii) and Theorem 1 too. ☐

4 A Berry-Esseen-type bound for nonstationary Gaussian subordinated triangular arrays

This section obtains a Berry-Esseen-type upper bound in the CLT (3.7) for non-stationary Gaussian subordinated triangular arrays following the method and results presented in Nourdin *et al.* (2011). We will refer NPP to the last paper in the rest of this section. To simplify the discussion, we restrict our task to the case when the functions $f_{k,n} = f$ in Theorem 1 (iii) do not depend on k, n. As in NPP, our starting point is the Hermite expansion (2.1) written as

$$f = \sum_{\ell=m}^{\infty} f_{(\ell)}, \qquad f_{(\ell)} := \sum_{|\boldsymbol{k}|=\ell} J_f(\boldsymbol{k})H_{\boldsymbol{k}}/\boldsymbol{k}!. \tag{4.1}$$

Following NPP and using the Hermite expansion in (4.1), we first define the following quantities: for $j\in\mathbb{Z}$, $\ell\ge m$, $N\ge m$, $n\in\mathbb{N}^*$ and $J\in\{1,\dots,n\}$:

$$\theta(j) \;:=\; |\rho(j)|\,, \quad K := \inf\{k\in\mathbb{N} : \theta(j) \le \frac{1}{\nu}, \forall |j|\ge k\}, \quad \theta := \sum_{j\in\mathbb{Z}}\theta(j)^m, \tag{4.2}$$

$$\sigma^2_{\ell,n} \;:=\; n^{-1}\sum_{t,t'=-n}^{n}\mathrm{Cov}\big(f_{(\ell)}(\boldsymbol{X}_n(t))\,, f_{(\ell)}(\boldsymbol{X}_n(t'))\big), \tag{4.3}$$

$$\gamma_{n,\ell,e} \;:=\; \frac{1}{n^{1/2}}\Big(2\theta\sum_{|j|\le n}\theta(j)^e\sum_{|j'|\le n}\theta(j')^{\ell-e}\Big)^{1/2} \quad (\text{for } 1\le e\le \ell-1), \tag{4.4}$$

$$A_{2,N} \;:=\; 2(2K+\nu^m\theta)\Big(\mathrm{E}[f^2(\boldsymbol{X})]\sum_{\ell=N+1}^{\infty}\mathrm{E}[f^2_{(\ell)}(\boldsymbol{X})]\Big)^{1/2}, \tag{4.5}$$

$$A_{3,n,N} := \frac{1}{2}\mathrm{E}[f^2(\boldsymbol{X})]\sum_{\ell=m}^{N}\Big(\frac{\nu^\ell}{\ell\ell!}\sum_{j=1}^{\ell-1} jj!\binom{\ell}{j}^2\sqrt{(2\ell-2j)!}\gamma_{n,\ell,j}\Big), \tag{4.6}$$

$$A_{4,n,N} := \frac{1}{2}\,\mathrm{E}[f^2(\boldsymbol{X})]\sum_{m\le \ell<\ell'\le N}\nu^{\ell'/2}\sqrt{\frac{\ell'!}{\ell!}}\frac{\ell+\ell'}{\ell}\binom{\ell'-1}{\ell-1}\big((\ell'-\ell)!\gamma_{n,\ell',\ell'-\ell}\big)^{1/2}, \tag{4.7}$$

$$A_{5,n,N} := \frac{\mathrm{E}[f^2(\boldsymbol{X})]}{2\sqrt{2}}\sum_{m\le\ell<\ell'\le N}(\ell+\ell')\sum_{j=1}^{\ell-1}(j-1)!\binom{\ell-1}{j-1}\binom{\ell'-1}{j-1}\sqrt{(\ell+\ell'-2j)!}\Big(\frac{\nu^\ell}{\ell!}\gamma_{n,\ell,\ell-j}+\frac{\nu^{\ell'}}{\ell'!}\gamma_{n,\ell',\ell'-j}\Big), \tag{4.8}$$

$$A_{6,n,J} := \frac{1}{2}|\partial f|^2_\infty \sup_{0\le\tau\le1}\sum_{|j|\le J}\Big\|\mathrm{E}[\boldsymbol{X}_n([n\tau])\boldsymbol{X}_n^\intercal([n\tau]+j)] - \mathrm{E}[\boldsymbol{W}_\tau(0)\boldsymbol{W}_\tau^\intercal(j)]\Big\|, \tag{4.9}$$

$$A_{7,J} := \frac{1}{2}\mathrm{E}\big[f^2(\boldsymbol{X})\big]\nu^m\sum_{|k|>J}\theta^m(k). \tag{4.10}$$

Note that terms $A_{2,n}, A_{3,n,N}$ and $A_{5,n,N}$ are the same as in NPP, $A_{4,n,N}$ is a minor improvement of the corresponding term in NPP, and $A_{6,n,J}$ reflects the "convergence rate" in (3.8). Term $A_{1,n}$ of NPP (which does not appear in our bounds) is "absorbed" in the term $\inf_{1\le J\le n} A_{7,J}$ in the bounds (i)-(iii), below, due to a somewhat a different approximation (see (4.15)).

Proposition 2 *Let the assumptions of Theorem 1 (iii) prevail, with $f_{k,n}\equiv f$ for all $1\le k\le n$, $n\in\mathbb{N}^*$ where $f:\mathbb{R}^\nu\to\mathbb{R}$ is a Lipschitz function with $|f(x)-f(y)|\le|\partial f|_\infty|x-y|$ for all $x\ne y\in\mathbb{R}^\nu$. Define $S_n := n^{-1/2}\sum_{t=1}^n f(\boldsymbol{X}_n(t))$ and let S be a zero-mean Gaussian random variable with a variance $\sigma_S^2 := \int_0^1\sum_{j\in\mathbb{Z}}\mathrm{Cov}\big(f(\boldsymbol{W}_\tau(0)), f(\boldsymbol{W}_\tau(j))\big)\,\mathrm{d}\tau<\infty$. Then:*

(i) For any function h twice continuously differentiable with bounded second derivative and for every $n>K$,

$$\Big|\mathrm{E}\big[h(S_n)\big]-\mathrm{E}\big[h(S)\big]\Big| \le |h''|_\infty\Big(\inf_{N\ge m}\big\{A_{2,N}+A_{3,n,N}+A_{4,n,N}+A_{5,n,N}\big\}+\inf_{1\le J\le n}\big\{A_{6,n,J}+A_{7,J}\big\}\Big). \tag{4.11}$$

(ii) For any Lipschitz function h, and for every $n>K$,

$$\begin{aligned}\Big|\mathrm{E}\big[h(S_n)\big]-\mathrm{E}\big[h(S)\big]\Big| &\le |h'|_\infty\Big(\frac{2}{\sigma_S}\inf_{1\le J\le n}\Big\{A_{6,n,J}+A_{7,J}\Big\}\\ &+\inf_{N\ge m}\Big\{\Big(\frac{1}{2\sigma_S}+\frac{1}{\big((2K+\nu^m)\mathrm{E}[f^2(\boldsymbol{X})]\big)^{1/2}}\Big)A_{2,N}+\frac{A_{3,n,N}+A_{4,n,N}+A_{5,n,N}}{\big(\sum_{\ell=m}^N\sigma^2_{\ell,n}\big)^{1/2}}\Big\}\Big).\end{aligned} \tag{4.12}$$

(iii) For any $z\in\mathbb{R}$, and for every $n>K$,

$$\begin{aligned}\big|\mathbb{P}(S_n\le z)]-\mathbb{P}(S\le z)\big| &\le \frac{2}{\sigma_S}\Big(\frac{2}{\sigma_S}\inf_{1\le J\le n}\Big\{A_{6,n,J}+A_{7,J}\Big\}\\ &+\inf_{N\ge m}\Big\{\Big(\frac{1}{2\sigma_S}+\frac{1}{\big((2K+\nu^m)\mathrm{E}[f^2(\boldsymbol{X})]\big)^{1/2}}\Big)A_{2,N}+\frac{A_{3,n,N}+A_{4,n,N}+A_{5,n,N}}{\big(\sum_{\ell=m}^N\sigma^2_{\ell,n}\big)^{1/2}}\Big\}\Big)^{1/2}.\end{aligned} \tag{4.13}$$

Proof of Proposition 4. Let us introduce a similar notation to NPP. Consider the Hilbert space $\mathfrak{H}=\mathbb{R}^{n\nu}$ with elements $u=(u_{t,l}, 1\le t\le n, 1\le l\le\nu)\in\mathfrak{H}$ and the scalar product $\langle u_{t,j},u_{t',j'}\rangle_{\mathfrak{H}} := \mathrm{E}X_n^{(j)}(t)X_n^{(j')}(t') = r_n^{(j,j')}(t,t')$. The $\ell-$fold tensor product and the symmetrized tensor product of $\mathfrak{H}$ are denoted by $\mathfrak{H}^{\otimes\ell}$ and $\mathfrak{H}^{\odot\ell}$, respectively. Let $\mathbb{L}^2(\mathfrak{X}_n)$ denote the space of r.v.'s subordinated to the Gaussian vector $\mathfrak{X}_n := (\boldsymbol{X}_n(t))_{1\le t\le n}$. Any element $\xi\in\mathbb{L}^2(\mathfrak{X}_n)$ admits a chaotic expansion $\xi=\sum_{\ell=0}^\infty I_{(\ell)}(g_{(\ell)})$, where $g_{(\ell)}\in\mathfrak{H}^{\otimes\ell}$ and the linear mapping $I_{(\ell)}:\mathfrak{H}^{\otimes\ell}\to\mathbb{L}^2(\mathfrak{X}_n)$ satisfies $I_{(\ell)}(g)=I_{(\ell)}(\mathrm{sym}(g))$, $\mathrm{E}I^2_{(\ell)}(g)=\ell!\|\mathrm{sym}(g)\|_{\mathfrak{H}^{\otimes\ell}}$, and $\mathrm{E}[I_{(\ell)}(g)I_{(\ell')}(g')]=0$, $\ell\ne\ell'$, $g_{(\ell)}\in\mathfrak{H}^{\otimes\ell}$, $g_{(\ell')}\in\mathfrak{H}^{\otimes\ell'}$, where sym denotes the symmetrization operator. In particular, for any $t=1,\dots,n$, $\boldsymbol{k}\in\mathbb{Z}_+^\nu$, $|\boldsymbol{k}|=:\ell$ we have $H_{\boldsymbol{k}}(\boldsymbol{X}_n(t))=I_{(\ell)}\big(g_\ell(\boldsymbol{k})\big)$, where

$$g_\ell(\boldsymbol{k}) := \mathrm{sym}\big(u_{t,1}^{\otimes k^{(1)}}\otimes\cdots\otimes u_{t,\nu}^{\otimes k^{(\nu)}}\big) = \sum_{\boldsymbol{v}\in\{1,\dots,\nu\}^\ell} b(\boldsymbol{v};\boldsymbol{k})\,u_{t,v_1}\otimes\cdots\otimes u_{t,v_\ell}$$

and where $b(\boldsymbol{v};\boldsymbol{k}) = \mathrm{sym}[\tilde{b}(\boldsymbol{v};\boldsymbol{k})]$ is the symmetrization of the function $\{1,\dots,\nu\}^\ell \ni \boldsymbol{v} = (v_1,\dots,v_\ell) \mapsto \tilde{b}(\boldsymbol{v};\boldsymbol{k}) := \prod_{r=1}^{\nu} \mathbf{1}\big(v_i = r, k_1+\dots+k_{r-1} < i \le k_1+\dots+k_r\big)$. Thus, $S_n = n^{-1/2}\sum_{t=1}^n f(\boldsymbol{X}_n(t))$ admits the chaotic expansion

$$S_n = \sum_{\ell=m}^{\infty} I_{(\ell)}(g_\ell^n) \quad \text{with} \quad g_\ell^n := \frac{1}{\sqrt{n}}\sum_{t=1}^n \sum_{\boldsymbol{v}\in\{1,\dots,\nu\}^\ell} b_\ell(\boldsymbol{v})\, u_{t,v_1}\otimes\cdots\otimes u_{t,v_\ell},$$

where $b_\ell(\boldsymbol{v}) := \sum_{|\boldsymbol{k}|=\ell}(J_f(\boldsymbol{k})/\boldsymbol{k}!)b(\boldsymbol{v};\boldsymbol{k})$ depend only on $f \in \mathbb{L}^2(\boldsymbol{X}_n(t)) = \mathbb{L}^2(\boldsymbol{X})$ and satisfy $\mathrm{E}f_{(\ell)}^2(\boldsymbol{X}) = \ell!\sum_{\boldsymbol{v}\in\{1,\dots,\nu\}^\ell} b_\ell^2(\boldsymbol{v})$, as in NPP. It is important that here the g_ℓ^n's are symmetric since the $b_\ell(\boldsymbol{v})$'s are symmetric. Therefore $\mathrm{E}I_{(\ell)}^2(g_\ell^n) = \ell!\|g_\ell^n\|^2_{\mathfrak{H}^{\otimes\ell}}$. Next, for $N \ge m$ consider the truncated expansion

$$S_{n,N} := \sum_{\ell=m}^{N} I_{(\ell)}(g_\ell^n).$$

Note that

$$\begin{aligned}
\mathrm{E}S_{n,N}^2 &= \sum_{\ell=m}^N \mathrm{E}I_{(\ell)}^2(g_\ell^n) = \sum_{\ell=m}^N \ell!\|g_\ell^n\|^2_{\mathfrak{H}^{\otimes\ell}} \\
&= \frac{1}{n}\sum_{\ell=m}^N \ell! \sum_{t,t'=1}^n \sum_{\boldsymbol{v},\boldsymbol{v}'\in\{1,\dots,\nu\}^\ell} b_\ell(\boldsymbol{v})b_\ell(\boldsymbol{v}')\,\langle u_{t,v_1}\otimes\cdots\otimes u_{t,v_\ell}, u_{t',v_1'}\otimes\cdots\otimes u_{t',v_\ell'}\rangle_{\mathfrak{H}^{\otimes\ell}} \\
&= \frac{1}{n}\sum_{\ell=m}^N \ell! \sum_{t,t'=1}^n \sum_{\boldsymbol{v},\boldsymbol{v}'\in\{1,\dots,\nu\}^\ell} b_\ell(\boldsymbol{v})b_\ell(\boldsymbol{v}')\prod_{i=1}^{\ell} r_n^{(v_i,v_i')}(t,t').
\end{aligned}$$

Using $|r_n^{(j,j')}(t,t')| \le \theta(t-t')$ similarly as in NPP we obtain

$$\Big|\mathrm{E}\big[h(S_n)\big] - \mathrm{E}\big[h(S_{n,N})\big]\Big| \le \frac{3}{2}\,(2K+\nu^m\theta)\,|h''|_\infty \Big(\mathrm{E}[f^2(\boldsymbol{X})]\sum_{\ell=N+1}^{\infty}\mathrm{E}[f_{(\ell)}^2(\boldsymbol{X})]\Big)^{1/2} \le \frac{3}{4}\,|h''|_\infty\, A_{2,N}. \tag{4.14}$$

For $N \ge m$, let $Z_{n,N}$ be a centered Gaussian random variable with variance $\mathrm{E}S_{n,N}^2 = \sum_{\ell=m}^N \sigma_{\ell,n}^2$, with $\sigma_{\ell,n}^2$ defined in (4.3). (Note that the last variance is slightly different from the variance of Z_N in (NPP, sec. 4.2).) Let D denote the Malliavin derivative in $\mathbb{L}^2(\mathfrak{X}_n)$, see NPP. Using $\ell^{-1}\mathrm{E}\|DI_{(\ell)}(g_\ell^n)\|^2_{\mathfrak{H}} = \ell!\|g_\ell^n\|^2_{\mathfrak{H}^{\otimes\ell}} = \sigma_{\ell,n}^2$, see (4.14), as in (NPP, (4.46)) we obtain

$$\begin{aligned}
\Big|\mathrm{E}\big[h(Z_{n,N})\big] - \mathrm{E}\big[h(S_{n,N})\big]\Big| &\le \frac{1}{2}\,|h''|_\infty \sum_{\ell,\ell'=m}^{N} \big\|\delta_{\ell\ell'}\sigma_{\ell,n}^2 - \ell^{-1}\langle DI_{(\ell)}(g_\ell^n), DI_{(\ell')}(g_{\ell'}^n)\rangle_{\mathfrak{H}}\big\|_{\mathbb{L}^2(\mathrm{P})} \\
&\le |h''|_\infty\big(A_{3,n,N} + A_{4,n,N} + A_{5,n,N}\big).
\end{aligned} \tag{4.15}$$

Next, using (NPP, (3.39))

$$\Big|\mathrm{E}\big[h(Z_{n,N})\big] - \mathrm{E}\big[h(S)\big]\Big| \le \frac{1}{2}\,|h''|_\infty\Big|\sum_{\ell=m}^N \sigma_{\ell,n}^2 - \sigma_S^2\Big| \le \frac{1}{2}\,|h''|_\infty\Big(\big|\sigma_n^2 - \sigma_S^2\big| + \Big|\sigma_n^2 - \sum_{\ell=m}^N \sigma_{\ell,n}^2\Big|\Big).$$

To estimate the difference $\sigma_n^2 - \sigma_S^2$, we use an interpolation identity from Houdré *et al.* (1998). Let $(\boldsymbol{X}_1,\boldsymbol{X}_2), (\boldsymbol{W}_1,\boldsymbol{W}_2)$ be two $(2\nu)-$dimensional Gaussian vectors with zero means and respective covariance matrices $\mathrm{E}[\boldsymbol{X}_i\boldsymbol{X}_i^\intercal] = \mathrm{E}[\boldsymbol{W}_i\boldsymbol{W}_i^\intercal] = I$, $i = 1,2$, $\mathrm{E}[\boldsymbol{X}_1\boldsymbol{X}_2^\intercal] = \Sigma_1$, $\mathrm{E}[\boldsymbol{W}_1\boldsymbol{W}_2^\intercal] = \Sigma_0$. For $\alpha \in [0,1]$ let $(\boldsymbol{X}_{1\alpha},\boldsymbol{X}_{2\alpha})$ denote the "interpolated" Gaussian vector with zero mean and $\mathrm{E}[\boldsymbol{X}_{i\alpha}\boldsymbol{X}_{i\alpha}^\intercal] = I$, $i = 1,2$, $\mathrm{E}[\boldsymbol{X}_{1\alpha}\boldsymbol{X}_{2\alpha}^\intercal] = (1-\alpha)\Sigma_0 + \alpha\Sigma_1$. Let $f \in \mathbb{L}^2(\boldsymbol{X})$ be a real function satisfying the conditions of Proposition 4. Then from ([16], (1.1), (1.3)) we obtain

$$\begin{aligned}
\big|\mathrm{Cov}\big(f(\boldsymbol{X}_1), f(\boldsymbol{X}_2)\big) - \mathrm{Cov}\big(f(\boldsymbol{W}_1), f(\boldsymbol{W}_2)\big)\big| &= \Big|\int_0^1 \mathrm{E}\left[\partial f(\boldsymbol{X}_{1\alpha})^\intercal(\Sigma_1-\Sigma_0)\partial f(\boldsymbol{X}_{2\alpha})\right]\mathrm{d}\alpha\Big| \\
&\le |\partial f|^2_\infty\,\|\Sigma_1 - \Sigma_0\|,
\end{aligned} \tag{4.16}$$

where $\partial f = (\partial f/\partial x^{(1)}, \ldots, \partial f/\partial x^{(\nu)})^\intercal \in \mathbb{R}^\nu$. Let $F_n(\tau) := \sum_{t'=1}^n \mathrm{Cov}\big(f(\boldsymbol{X}_n([n\tau])), f(\boldsymbol{X}_n(t'))\big)$, $\tau \in [0,1]$ so that $\sigma_n^2 = \int_0^1 F_n(\tau)\mathrm{d}\tau$. Using (4.16), for $1 \le J \le n$ we can write $\big|\sigma_n^2 - \sigma_S^2\big| \le R_1(n,J) + R_2(n,J)$, where

$$\begin{aligned} R_1(n,J) \quad &:= \quad \int_0^1 \sum_{|j|\le J} \Big|\mathrm{Cov}\big(f\big(\boldsymbol{X}_n([n\tau])\big), f\big(\boldsymbol{X}_n([n\tau]+j)\big) - \mathrm{Cov}\big(f(\boldsymbol{W}_\tau(0)), f(\boldsymbol{W}_\tau(j))\big)\Big|\mathrm{d}\tau \ \le\ 2A_{6,n,J}, \\ R_2(n,J) \quad &\le \quad 2\,\mathrm{E}\big[f^2(\boldsymbol{X})\big]\,\nu^m \sum_{|k|>J} \theta^m(k) \ = \ 2A_{7,J}. \end{aligned}$$

We also have $\big|\sigma_n^2 - \sum_{\ell=m}^N \sigma_{\ell,n}^2\big| = \sum_{\ell=N+1}^\infty \sigma_{\ell,n}^2 \ \le\ \frac{1}{2} A_{2,N}$, as in (4.14). Therefore, $\Big|\sum_{\ell=m}^N \sigma_{\ell,n}^2 - \sigma_S^2\Big| \le 2A_{6,n,J} + 2A_{7,J} + \frac{1}{2} A_{2,N}$, implying

$$\Big|\mathrm{E}\big[h(Z_{n,N})\big] - \mathrm{E}\big[h(S)\big]\Big| \quad \le \quad |h''|_\infty\Big(A_{6,n,J} + A_{7,J} + \frac{1}{4} A_{2,N}\Big) \qquad \text{for} \quad 1 \le J \le n. \tag{4.17}$$

Finally combining (4.14), ((4.15), and (4.17) results in

$$\begin{aligned} \Big|\mathrm{E}\big[h(S_n)\big] - \mathrm{E}\big[h(S)\big]\Big| \quad &\le \quad |h''|_\infty\Big(A_{2,N} + A_{3,n,N} + A_{4,n,N} + A_{5,n,N} + \inf_{1\le J\le n}(A_{6,n,J} + A_{7,J})\Big) \\ &\le \quad |h''|_\infty\Big(\inf_{N\ge m}\big\{A_{2,N} + A_{3,n,N} + A_{4,n,N} + A_{5,n,N}\big\} + \inf_{1\le J\le n}\big\{A_{6,n,J} + A_{7,J}\big\}\Big), \end{aligned}$$

proving the bound in (4.11).

(ii) Following (NPP, proof of Theorem 2.2-(2)) and the previous results, for a Lipschitz function h we obtain:

$$\begin{aligned} \big|\mathrm{E}\big[h(S_n)\big] - \mathrm{E}\big[h(S_{n,N})\big]\big| \quad &\le \quad |h'|_\infty\big((2K + \nu^m)\mathrm{E}[f^2(\boldsymbol{X})]\big)^{-1/2} A_{2,N}, \\ \big|\mathrm{E}\big[h(Z_{n,N})\big] - \mathrm{E}\big[h(S_{n,N})\big]\big| \quad &\le \quad 2\,|h'|_\infty\,\Big(\sum_{\ell=m}^N \sigma_{\ell,n}^2\Big)^{-1/2}\big(A_{3,n,N} + A_{4,n,N} + A_{5,n,N}\big) \\ \text{and} \quad \big|\mathrm{E}\big[h(Z_{n,N})\big] - \mathrm{E}\big[h(S)\big]\big| \quad &\le \quad \frac{|h'|_\infty}{\sigma_S}\Big(\frac{1}{2} A_{2,N} + \inf_{1\le J\le n}(A_{6,n,J} + A_{7,J})\Big) \end{aligned}$$

and therefore (4.12) is established.

(iii) Bound (4.13) is obtained exactly as in (NPP, proof of Theorem 2.2-(3)). $\square$

5 Applications of Lemma 1 and Theorem 1

5.1 Application to the IR statistic

This application was developed in Bardet and Surgailis (2011, 2012). Let $(X_t)_{t\in[0,1]}$ be a continuous time Gaussian process with zero mean and generally nonstationary increments locally resembling a fractional Brownian motion with Hurst parameter $H(t) \in (0,1)$. Consider the Increment Ratio (IR) statistic

$$R^{2,n}(X) \quad := \quad \frac{1}{n-2}\sum_{k=0}^{n-3} \frac{|\Delta_k^{2,n}X + \Delta_{k+1}^{2,n}X|}{|\Delta_k^{2,n}X| + |\Delta_{k+1}^{2,n}X|},$$

with $\Delta_k^{2,n}X = X_{(k+2)/n} - 2\,X_{(k+1)/n} + X_{k/n}$ and the convention $\frac{0}{0} := 1$. Let $\sigma_{2,n}^2(k) := \mathrm{E}\Big[\Big(\Delta_k^{2,n}X\Big)^2\Big]$ and

$$Y_n^{(1)}(k) \quad := \quad \frac{\Delta_k^{2,n}X}{\sigma_{2,n}(k)}, \qquad Y_n^{(2)}(k) \quad := \quad \frac{\Delta_{k+1}^{2,n}X}{\sigma_{2,n}(k)}.$$

Then $R^{2,n}(X) = \frac{1}{n-2}\sum_{k=0}^{n-3} f(\boldsymbol{Y}_n(k))$, $f(x^{(1)}, x^{(2)}) := |x^{(1)} + x^{(2)}|/(|x^{(1)}| + |x^{(2)}|)$ can be written as the sum of nonlinear function f of Gaussian vectors $\boldsymbol{Y}_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k)) \in \mathbb{R}^2$, $0 \le k \le n-3$. These

Gaussian vectors can be standardized, leading to the expression $R^{2,n}(X) = \frac{1}{n-2}\sum_{k=0}^{n-3} f_{n,k}\left(\boldsymbol{X}_n(k)\right)$ of the IR statistics as the sum of some functions $f_{n,k}$ of standardized Gaussian vectors $\boldsymbol{X}_n(k)$, $0 \le k \le n-3$. (It is easy to check that the centered functions $f_{n,k} - \mathrm{E}[f_{n,k}(\boldsymbol{X})]$ have the Hermite rank 2.) If (X_t) satisfies some additional conditions (specifying the decay rate of correlations of increments and the convergence rate to the tangent process), Theorem 1 can be applied to establish that $\sqrt{n}\big(R^{2,n}(X) - \int_0^1 \Lambda(H(t))\,\mathrm{d}t\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2)$ with an explicit function Λ and a variance σ^2. An application of Lemma 1 to bound the 4th moment $\mathrm{E}(R^{2,n}(X) - \mathrm{E}R^{2,n}(X))^4$ provides a crucial step in the proof of the almost sure consistency of the IR statistic, *i.e.* $R^{2,n}(X) \xrightarrow[n\to\infty]{a.s.} \int_0^1 \Lambda(H(t))\,\mathrm{d}t$. See Bardet and Surgailis (2011) for details. Local versions of the IR statistic for point-wise estimation of $H(t)$ are developed in Bardet and Surgailis (2012). The study of the asymptotic properties of these estimators in the last paper is also based on Theorem 1 and Lemma 1.

5.2 A central limit theorem for functions of locally stationary Gaussian processes

Using an adaptation of Dahlhaus and Polonik (2006, 2009), we will say that $(X_{t,n})_{1\le t\le n,\, n\in\mathbb{N}^*}$ is a locally stationary Gaussian process if

$$X_{t,n} := \sum_{j\in\mathbb{Z}} a_{t,n}(j)\,\varepsilon_{t-j}, \qquad \text{for all } 1 \le t \le n,\ n \in \mathbb{N}^*, \tag{5.1}$$

where $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a sequence of independent standardized Gaussian variables and for $1 \le t \le n$, $n \in N^*$ the sequences $(a_{t,n}(j))_{j\in\mathbb{Z}}$ are such that there exist $K \ge 0$ and $\alpha < 1/2$ satisfying for all $n \in \mathbb{N}^*$ and $j \in \mathbb{Z}$,

$$\max_{1\le t\le n} |a_{t,n}(j)| \le \frac{K}{u_j}, \qquad \text{with } \ u_j := \max(1, |j|^{\alpha-1}) \ \text{ for } j \in \mathbb{Z} \tag{5.2}$$

and such that there exist functions $\tau \in (0,1] \mapsto a(\tau,j) \in \mathbb{R}$ satisfying the following conditions:

$$\sup_{\tau\in(0,1]} |a(\tau,j)| \quad \le \quad \frac{K}{u_j}, \qquad \forall\, j \in \mathbb{Z}, \tag{5.3}$$

$$\text{and} \qquad \sup_{\tau\in(0,1]} \max_{|[n\tau]-k|\le L} \big|(a_{k,n}(j) - a(\tau,j)\big| \quad \to \quad 0, \qquad \forall\, j \in \mathbb{Z}, \quad \forall\, L > 0. \tag{5.4}$$

For $\tau \in (0,1]$ introduce a stationary Gaussian process

$$W_\tau(t) := \sum_{j\in\mathbb{Z}} a(\tau,j)\,\varepsilon_{t-j}, \qquad t \in \mathbb{Z}.$$

with spectral density $g_\tau(v) = |\hat{a}(\tau,v)|^2$, $\hat{a}(\tau,v) := (2\pi)^{-1/2}\sum_{j\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}jv} a(\tau,j)$, $v \in [-\pi,\pi]$. Let

$$\boldsymbol{Y}_n(k) \ := \ \big(X_{k+1,n},\dots,X_{k+\nu,n}\big)^\intercal, \qquad \boldsymbol{W}_\tau(j) \ := \ \big(W_\tau(j+1),\dots,W_\tau(j+\nu)\big)^\intercal.$$

Note $(\boldsymbol{W}_\tau(j))_{j\in\mathbb{Z}}$ is a $\mathbb{R}^\nu-$valued stationary Gaussian process. Let

$$\Sigma_{k,n} \ := \ \mathrm{E}[\boldsymbol{Y}_n(k)\boldsymbol{Y}_n(k)^\intercal], \qquad \Sigma_\tau := \mathrm{E}[\boldsymbol{W}_\tau(0)\boldsymbol{W}_\tau(0)^\intercal].$$

Proposition 3 *In addition to (5.1) - (5.4), assume that*

$$\sup_{\tau\in(0,1]} \|\Sigma_\tau^{-1}\| < \infty. \tag{5.5}$$

Let $f_{k,n} \in \mathbb{L}_0^2(\boldsymbol{Y}_n(k))$, $1 \le k \le n, n \ge 1$ *be a triangular array of functions all having a generalized Hermite rank at least* $m > 1/(1-2\alpha)$. *Let there exists a* $\mathbb{L}_0^2(\boldsymbol{X})-$*valued continuous function* $\tilde{\phi}_\tau, \tau \in (0,1]$ *such that relation (3.12) holds, with* $\tilde{f}_{k,n}(\boldsymbol{x}) := f_{k,n}(\Sigma_{k,n}^{1/2}\boldsymbol{x})$. *Then the CLT of (3.13) holds, with*

$$\sigma^2 := \int_0^1 \mathrm{d}\tau \sum_{j\in\mathbb{Z}} \mathrm{E}\big[\phi_\tau\big(\boldsymbol{W}_\tau(0)\big)\,\phi_\tau\big(\boldsymbol{W}_\tau(j)\big)\big] \tag{5.6}$$

and $\phi_\tau(\boldsymbol{x}) := \tilde{\phi}_\tau(\Sigma_\tau^{-1/2}\boldsymbol{x})$ *defined as in Corollary 2.*

Proof. We apply Corollary 2. Let us first check

$$\sup_{\tau \in (0,1]} \|\Sigma_{[n\tau],n} - \Sigma_\tau\| \underset{n\to\infty}{\longrightarrow} 0. \tag{5.7}$$

We have

$$\big|\sigma_{[n\tau],n}(p,q) - \sigma_\tau(p,q)\big| \quad = \quad \Big|\sum_{j\in\mathbb{Z}} \big(a_{[n\tau]+p,n}(p+j)a_{[n\tau]+q,n}(q+j) - a(\tau,p+j)a(\tau,q+j)\big)\Big| \ \le \ T_{n,J} + T''_{n,J},$$

where

$$T'_{n,J} \quad := \quad 2K^2 \sum_{|j|>J} u_{p+j}u_{q+j}, \qquad T''_{n,J} := \sum_{|j|\le J} \big|a_{[n\tau]+p,n}(p+j)a_{[n\tau]+q,n}(q+j) - a(\tau,p+j)a(\tau,q+j)\big)\big|$$

according to (5.2) and (5.3). Clearly, $T'_{n,J}$ can be made arbitrarily small by choosing J large enough. Then for any $J<\infty$ fixed, we have that $\sup_{\tau\in(0,1]} T''_{n,J} \to 0$ according to assumption (5.4). This proves (5.7). In a similar way, one verify that for any $\tau \in (0,1]$, $j,j' \in \mathbb{Z}$, $\|\mathrm{E}[\boldsymbol{Y}_n([n\tau]+j)\boldsymbol{Y}_n([n\tau]+j')^\intercal] - \mathrm{E}[\boldsymbol{W}_\tau(j)\boldsymbol{W}_\tau(j')^\intercal]\| \underset{n\to\infty}{\longrightarrow} 0$ implying condition (3.8). The dominating condition (3.3) on cross-covariances is ensured by (5.2) and the fact that $(1-2\alpha)m > 1$. The remaining conditions of Corollary 2 are trivially satisfied. $\square$

Remark 2 Dahlhaus and Polonik (2006, 2009) discussed the short-memory case $(a_{t,n}(j))_{j\in\mathbb{Z}} \in \ell^1$, $1 \le t \le n$ only. On the other hand, condition (5.2) allows for the long-memory case $(a_{t,n}(j))_{j\in\mathbb{Z}} \in \ell^2$, $\sum_{j\in\mathbb{Z}} |a_{t,n}(j)| = \infty$. The last case is also discussed in Roueff and von Sachs (2010), where similar conditions as (5.2) and (5.3) are provided in spectral terms. It is not clear whether condition (5.4) allows for jumps of the parameter curves $\tau \mapsto a(\tau,\cdot)$ as in Dahlhaus and Polonik (2006, 2009), in particular, for abrupt changes of the memory intensity of Gaussian process (5.1). See also Lavancier *et al.* (2011) for a related class of nonstationary moving average processes with long memory.

Remark 3 Note that $\boldsymbol{x}^\intercal \Sigma_\tau \boldsymbol{x} = \int_{-\pi}^{\pi} g_\tau(v)\big|\sum_{j=1}^{\nu} \mathrm{e}^{\mathrm{i}jv} x^{(j)}\big|^2 \mathrm{d}v$ for any $\boldsymbol{x} = (x^{(1)},\dots,x^{(\nu)})^\intercal \in \mathbb{R}^\nu$. Therefore condition $\inf_{v\in[-\pi,\pi],\tau\in(0,1]} g_\tau(v) \ge \gamma > 0$ on the spectral density of $(W_\tau(t))$ implies condition (5.5), since $\boldsymbol{x}^\intercal \Sigma_\tau \boldsymbol{x} \ge c|\boldsymbol{x}|^2$, $c := 2\pi\nu\gamma > 0$.

Remark 4 For stationary Gaussian long memory process, condition $m(1-2\alpha) > 1$ was first obtained in Taqqu (1975). Proposition 3 can be applied to prove the asymptotic normality of various statistics of locally stationary processes, see, e.g., Roueff and von Sachs (2010).

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